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$E_{7(7)}$ Duality, BPS Black–Hole Evolution and Fixed Scalars *

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Abstract

We study the general equations determining BPS Black Holes by using a Solvable Lie Algebra representation for the homogenous scalar manifold U/H of extended supergravity. In particular we focus on the N=8 case and we perform a general group theoretical analysis of the Killing spinor equation enforcing the BPS condition. Its solutions parametrize the U–duality orbits of BPS solutions that are characterized by having 40 of the 70 scalars fixed to constant values. These scalars belong to hypermultiplets in the $N = 2$ decomposition of the $N = 8$ theory. Indeed it is shown that those decompositions of the Solvable Lie algebra into appropriate subalgebras which are enforced by the existence of BPS black holes are the same that single out consistent truncations of the N=8 theory to intereacting theories with lower supersymmetry. As an exemplification of the method we consider the simplified case where the only non-zero fields are in the Cartan subalgebra $\mathcal{H} \subset Solv(U/H)$ and correspond to the radii of string toroidal compactification. Here we derive and solve the mixed system of first and second order non linear differential equations obeyed by the metric and by the scalar fields. So doing we retrieve the generating solutions of heterotic black holes with two charges. Finally, we show that the general $N = 8$ generating solution is based on the 6 dimensional solvable subalgebra $Solv\left[(SL(2,\mathbb{R})/U(1))^3\right]$.

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1 Introduction

Interest in the extremal black hole solutions of $D = 4$ supergravity theories has been quite vivid in the last couple of years [1, 2] and it is just part of a more general interest in the p–brane classical solutions of supergravity theories in all dimensions $4 \leq D \leq 11$ [3, 4]. This interest streams from the interpretation of the classical solutions of supergravity that preserve a fraction of the original supersymmetries as the BPS non perturbative states necessary to complete the perturbative string spectrum and make it invariant under the many conjectured duality symmetries [5, 6, 7, 8, 9]. This identification has become quite circumstantial with the advent of D–branes [10] and the possibility raised by them of a direct construction of the BPS states within the language of perturbative string theory extended by the choice of Dirichlet boundary conditions [11].

Basic feature of the non–perturbative states of the string spectrum is that they can carry Ramond–Ramond charges forbidden at the perturbative level. On the other hand, the important observation by Hull and Townsend [8, 7] is that at the level of the low energy supergravity lagrangians all fields of both the Neveu-Schwarz,Neveu-Schwarz (NS-NS) and the Ramond-Ramond (R-R) sector are unified by the group of duality transformations U which is also the isometry group of the homogenous scalar manifold $\mathcal{M}_{scalar} = U/H$. At least this is true in theories with sufficiently large number of supersymmetries that is with $N \geq 3$ in $D = 4$ or, in a dimensional reduction invariant language with #supercharges $\geq$ 12. This points out that the distinction between R-R and NS-NS sectors is just an artifact of perturbative string theory. It also points out to the fact that the unifying symmetry between the perturbative and non-perturbative sectors is already known from supergravity, namely it is the U-duality group. Indeed the basic conjecture of Hull and Townsend is that the restriction to integers $U(\mathbb{Z})$ of the U Lie group determined by supergravity should be an exact symmetry of non-perturbative string theory.

These observations raise the question of the group-theoretical basis for the separation of the string spectrum, in particular the light modes yielding the field content of supergravity, into a NS-NS and a R-R sector. This question was answered by the present authors [12, 13] using the solvable Lie algebra representation of the scalar field sector.

In the present paper we want to consider the application of solvable Lie algebras to the derivation of the differential equations that characterize BPS states as classical supergravity solutions.

From an abstract viewpoint BPS saturated states are characterized by the fact that they preserve, in modern parlance, 1/2 (or 1/4, or 1/8) of the original supersymmetries. What this actually means is that there is a suitable projection operator $\mathbb{P}_{BPS}^2 = \mathbb{P}_{BPS}$ acting on the supersymmetry charge Q_{SUSY}, such that:

$$(\mathbb{P}_{BPS}\, Q_{SUSY}) \mid \text{BPS state} >= 0 \tag{1}$$

Since the supersymmetry transformation rules of any supersymmetric field theory are linear in the first derivatives of the fields eq.(1) is actually a *system of first order differential equations.* This system has to be combined with the second order field equations of supergravity and the common solutions to both system of equations is a classical BPS saturated state. That it is

actually an exact state of non–perturbative string theory follows from supersymmetry representation theory. The classical BPS state is by definition an element of a *short supermultiplet* and, if supersymmetry is unbroken, it cannot be renormalized to a *long supermultiplet*.

Translating eq. (1) into an explicit first order differential system requires knowledge of the supersymmetry transformation rules of supergravity and it is at this level that solvable Lie algebras can play an important role. In order to grasp the significance of the above statement let us first rapidly review, as an example, the algebraic definition of $D = 4$, $N = 2\nu$ BPS states and then the idea of the solvable Lie algebra representation of the scalar sector.

The $D = 4$ supersymmetry algebra with an even number $N = 2\nu$ of supersymmetry charges can be written in the following form:

$$\left\{\overline{Q}_{aI|\alpha}\,,\,\overline{Q}_{bJ|\beta}\right\} = \mathrm{i}\,(C\,\gamma^{\mu})_{\alpha\beta}\,P_{\mu}\,\delta_{ab}\,\delta_{IJ} - C_{\alpha\beta}\,\epsilon_{ab}\times\mathbb{Z}_{IJ}$$
$$(a,b=1,2 \qquad ; \qquad I,J=1,\ldots,\nu) \tag{2}$$

where the SUSY charges $\overline{Q}_{aI} \equiv Q^{\dagger}_{aI}\gamma_0 = Q^T_{ai}\,C$ are Majorana spinors, C is the charge conjugation matrix, P_μ is the 4–momentum operator, ϵ_{ab} is the two–dimensional Levi Civita symbol and the symmetric tensor $\mathbb{Z}_{IJ} = \mathbb{Z}_{JI}$ is the central charge operator. It can always be diagonalized $\mathbb{Z}_{IJ} = \delta_{IJ}\,Z_J$ and its ν eigenvalues Z_J are the central charges.

The Bogomolny bound on the mass of a generalized monopole state:

$$M \geq |\,Z_I| \qquad \forall Z_I\,,\, I=1,\ldots,\nu \tag{3}$$

is an elementary consequence of the supersymmetry algebra and of the identification between *central charges* and *topological charges*. To see this it is convenient to introduce the following reduced supercharges:

$$\overline{S}^{\pm}_{aI|\alpha} = \frac{1}{2}\left(\overline{Q}_{aI}\gamma_0 \pm \mathrm{i}\,\epsilon_{ab}\,\overline{Q}_{bI}\right)_{\alpha} \tag{4}$$

They can be regarded as the result of applying a projection operator to the supersymmetry charges:

$$\begin{aligned}\overline{S}^{\pm}_{aI} &= \overline{Q}_{bI}\,\mathbb{P}^{\pm}_{ba}\\ \mathbb{P}^{\pm}_{ba} &= \frac{1}{2}\,(\mathbf{1}\delta_{ba} \pm \mathrm{i}\epsilon_{ba}\gamma_0)\end{aligned} \tag{5}$$

Combining eq.(2) with the definition (4) and choosing the rest frame where the four momentum is $P_\mu = (M,0,0,0)$, we obtain the algebra:

$$\left\{\overline{S}^{\pm}_{aI}\,,\,\overline{S}^{\pm}_{bJ}\right\} = \pm\epsilon_{ac}\,C\,\mathbb{P}^{\pm}_{cb}\,(M \mp Z_I)\,\delta_{IJ} \tag{6}$$

By positivity of the operator $\left\{S^{\pm}_{aI}\,,\,\overline{S}^{\pm}_{bJ}\right\}$ it follows that on a generic state the Bogomolny bound (3) is fulfilled. Furthermore it also follows that the states which saturate the bounds:

$$(M \pm Z_I)\,|\text{BPS state},i\rangle = 0 \tag{7}$$

are those which are annihilated by the corresponding reduced supercharges:

$$\overline{S}^{\pm}_{aI}\,|\text{BPS state},i\rangle = 0 \tag{8}$$

On one hand eq.(8) defines *short multiplet representations* of the original algebra (2) in the following sense: one constructs a linear representation of (2) where all states are identically annihilated by the operators $\overline{S}^{\pm}_{aI}$ for $I = 1, \ldots, n_{max}$. If $n_{max} = 1$ we have the minimum shortening, if $n_{max} = \nu$ we have the maximum shortening. On the other hand eq.(8) can be translated into a first order differential equation on the bosonic fields of supergravity.

Indeed, let us consider a configuration where all the fermionic fields are zero. Setting the fermionic SUSY rules appropriate to such a background equal to zero we find the following Killing spinor equation:

$$0 = \delta \text{fermions} = \text{SUSY rule} \left(\text{bosons}, \epsilon_{AI}\right) \tag{9}$$

where the SUSY parameter satisfies the following conditions:

$$\begin{array}{rcll} \xi^\mu \, \gamma_\mu \, \epsilon_{aI} & = & \mathrm{i} \, \varepsilon_{ab} \, \epsilon^{bI} & ; \quad I = 1, \ldots, n_{max} \\ \epsilon_{aI} & = & 0 & ; \quad I > n_{max} \end{array} \tag{10}$$

Here ξ^μ is a time-like Killing vector for the space-time metric and $\epsilon_{aI}, \epsilon^{aI}$ denote the two chiral projections of a single Majorana spinor:

$$\gamma_5 \, \epsilon_{aI} \; = \; \epsilon_{aI} \quad ; \quad \gamma_5 \, \epsilon^{aI} \; = -\epsilon^{aI} \tag{11}$$

Eq.(9) has two features which we want to stress as main motivations for the developments presented in later sections:

1. It requires an efficient parametrization of the scalar field sector

2. It breaks the original $SU(2\nu)$ automorphism of the supersymmetry algebra to the subgroup $SU(2) \times SU(2\nu - 2) \times U(1)$

The first feature is the reason why the use of the solvable Lie algebra $Solv$ associated with $U/SU(2\nu) \times H'$ is of great help in this problem. The second feature is the reason why the solvable Lie algebra $Solv$ has to be decomposed in a way appropriate to the decomposition of the isotropy group $H = SU(2\nu) \times H'$ with respect to the subgroup $SU(2) \times SU(2\nu - 2) \times U(1) \times H'$.

To explain what is involved in the above statements, a quick review of the solvable Lie algebra representation will be given in section 5 (see also ref. [12, 13]).

1.1 N=2 decomposition in the $N = 8$ theory

Although our goal is that of developing general methods for the study of BPS p–branes in all higher dimensional supergravities, in the present paper we concentrate on the maximally extended four–dimensional theory, namely $N = 8$ supergravity. Hence the relevant U–duality group is $E_{7(7)}$ and the relevant solvable Lie algebra is that associated with the homogeneous manifold $E_{7(7)}/SU(8)$. Since it is maximally non compact the rank of $Solv\left(E_{7(7)}/SU(8)\right)$ is seven. Henceforth we introduce the notation:

$$Solv_7 \; \equiv \; Solv\left(E_{7(7)}/SU(8)\right) \tag{12}$$

According to the previous discussion the Killing spinor equation for $N = 8$ BPS states requires that $Solv_7$ should be decomposed according to the decomposition of the isotropy subgroup: $SU(8) \longrightarrow SU(2) \times U(6)$. We show in later sections that the corresponding decomposition of the solvable Lie algebra is the following one:

$$Solv_7 = Solv_3 \oplus Solv_4 \tag{13}$$

$$\begin{array}{rclcrcl} Solv_3 & \equiv & Solv\,(SO^\star(12)/U(6)) & \quad & Solv_4 & \equiv & Solv\left(E_{6(4)}/SU(2)\times SU(6)\right) \\ \text{rank } Solv_3 & = & 3 & & \text{rank } Solv_4 & = & 4 \\ \text{dim } Solv_3 & = & 30 & & \text{dim } Solv_4 & = & 40 \end{array} \tag{14}$$

The rank three Lie algebra $Solv_3$ defined above describes the thirty dimensional scalar sector of $N = 6$ supergravity, while the rank four solvable Lie algebra $Solv_4$ contains the remaining fourty scalars belonging to $N = 6$ spin 3/2 multiplets. It should be noted that, individually, both manifolds $\exp[Solv_3]$ and $\exp[Solv_4]$ have also an $N = 2$ interpretation since we have:

$$\begin{array}{rcl} \exp[Solv_3] & = & \text{homogeneous special Kähler} \\ \exp[Solv_4] & = & \text{homogeneous quaternionic} \end{array} \tag{15}$$

so that the first manifold can describe the interaction of 15 vector multiplets, while the second can describe the interaction of 10 hypermultiplets. Indeed if we decompose the $N = 8$ graviton multiplet in $N = 2$ representations we find:

$$\text{N=8}\,\textbf{spin 2} \stackrel{N=2}{\longrightarrow} \textbf{spin 2} + 6 \times \textbf{spin 3/2} + 15 \times \textbf{vect. mult.} + 10 \times \textbf{hypermult.} \tag{16}$$

Although at the level of linearized representations of supersymmetry we can just delete the 6 spin 3/2 multiplets and obtain a perfectly viable $N = 2$ field content, at the full interaction level this truncation is not consistent. Indeed, in order to get a consistent $N = 2$ truncation the complete scalar manifold must be the *direct product* of *a special Kähler* manifold with *a quaternionic manifold*. This is not true in our case since putting together $\exp[Solv_3]$ with $\exp[Solv_4]$ we reobtain the $N = 8$ scalar manifold $E_{7(7)}/SU(8)$ which is neither a direct product nor Kählerian, nor quaternionic. The blame for this can be put on the decomposition (13) which is a direct sum of vector spaces but not a direct sum of Lie algebras: in other words we have

$$[Solv_3\,,\,Solv_4] \neq 0 \tag{17}$$

The problem of deriving consistent $N = 2$ truncations is most efficiently addressed in the language of Alekseveeskian solvable algebras [15]. $Solv_3$ is a Kähler solvable Lie algebra, while $Solv_4$ is a quaternionic solvable Lie algebra. We must determine a Kähler subalgebra $\mathcal{K} \subset Solv_3$ and a quaternionic subalgebra $\mathcal{Q} \subset Solv_4$ in such a way that:

$$[\mathcal{K}\,,\,\mathcal{Q}] = 0 \tag{18}$$

Then the truncation to the vector multiplets described by $\mathcal{K}$ and the hypermultiplets described by $\mathcal{Q}$ is consistent at the interaction level. An obvious solution is to take no vector multiplets

($\mathcal{K} = 0$) and all hypermultiplets ($\mathcal{Q} = Solv_4$) or viceversa ($\mathcal{K} = Solv_3$), ($\mathcal{Q} = 0$). Less obvious is what happens if we introduce just one hypermultiplet, corresponding to the minimal one–dimensional quaternionic algebra. In later sections we show that in that case the maximal number of admitted vector multiplets is 9. The corresponding Kähler subalgebra is of rank 3 and it is given by:

$$Solv_3 \supset \mathcal{K}_3 \equiv Solv\,(SU(3,3)/SU(3)\times U(3)) \tag{19}$$

Note that, as we will discuss in the following, the 18 scalars parametrizing the manifold $SU(3,3)/SU(3)\times U(3)$ are all the scalars in the NS-NS sector of $SO^*(12)$. A thoroughful discussion of the N=2 truncation problem and of its solution in terms of solvable Lie algebra decompositions is discussed in section 6.1. At the level of the present introductory section we want to stress the relation of the decomposition (13) with the Killing spinor equation for BPS black-holes.

Indeed, as just pointed out, the decomposition (13) is implied by the $SU(2)\times U(6)$ covariance of the Killing spinor equation. As we show in section (3.2) this equation splits into various components corresponding to different $SU(2)\times U(6)$ irreducible representations. Introducing the decomposition (13) we will find that the 40 scalars belonging to $Solv_4$ are constants independent of the radial variable r. Only the 30 scalars in the Kähler algebra $Solv_3$ can have a radial dependence. In fact their radial dependence is governed by a first order differential equation that can be extracted from a suitable component of the Killing spinor equation. In this way we see that the same solvable Lie algebra decompositions occurring in the problem of N=2 truncations of N=8 supergravity occur also in the problem of constructing N=8 BPS black holes.

We present now our plan for the next sections.

In section 2 we discuss the structure of the scalar sector in $N = 8$ supergravity and its supersymmetry transformation rules. As just stated, our goal is to develop methods for the analysis of BPS states as classical solutions of supergravity theories in all dimensions and for all values of N. Many results exist in the literature for the $N = 2$ case in four dimensions [17, 18, 19], where the number of complex scalar fields involved just equals the number of differential equations one obtains from the Killing spinor condition. Our choice to focus on the $N = 8$, $D = 4$ case is motivated by the different group–theoretical structure of the Killing spinor equation in this case and by the fact that it is the maximally extended supersymmetric theory.

In section 3 we introduce the black–hole ansatz and we show how using roots and weights of the $E_{7(7)}$ Lie algebra we can rewrite in a very intrinsic way the Killing spinor equation. We analyse its components corresponding to irreducible representations of the isotropy subalgebra $U(1)\times SU(2)\times SU(6)$ and we show the main result, namely that the 40 scalars in the $Solv_4$ subalgebra are constants.

In section 4 we exemplify our method by explicitly solving the simplified model where the only non–zero fields are the dilatons in the Cartan subalgebra. In this way we retrieve the known a–model solutions of N=8 supergravity.

The following two sections 5 and 6 are concerned with the method and the results of our computer aided calculations on the embedding of the subalgebras $U(1)\times SU(2)\times SU(6) \subset SU(8)$

in $E_{7(7)}$ and with the structure of the solvable Lie algebra decomposition already introduced in eq.(13). In particular we study the problem of consistent N=2 truncations using Alekseevski formalism. These two sections, being rather technical, can be skipped in a first reading by the non interested reader. We note however that many of the results there obtained are used for the discussion of the subsequent section. In particular, these results are preliminary for the allied project of gauging the maximal gaugeable abelian ideal $\mathcal{G}_{abel}$, which we outlined in our earlier paper [13] and which will be postponed to a future publication. Note that such a gauging should on one hand produce spontaneous partial breaking of $N = 8$ supersymmetry and on the other hand be interpretable as due to the condensation of $N = 8$ BPS black-holes.

In the final section 7 we address the question of the most general BPS black-hole. Using the little group of the charge vector in its normal form which, following [21], is identified with $SO(4,4)$, we are able to conclude that the only relevant scalar fields are those associated with the solvable Lie subalgebra:

$$Solv\left(\frac{SL(2,\mathbb{R})^3}{U(1)^3}\right) \subset Solv_3 \tag{20}$$

where the 6 scalars that parametrize the manifold $\frac{SL(2,\mathbb{R})^3}{U(1)^3}$ are all in the NS-NS sector.

Moreover, with an appropriate identification, we show how the calculation of the fixed scalars performed by the authors of [20] in the N=2 STU model amounts to a solution of the same problem also in the N=8 theory. The most general solution can be actually generated by U-duality rotations of $E_{7(7)}$.

Therefore, the final result of our whole analysis, summarized in the conclusions is that up to U-duality transformations the most general $N = 8$ black-hole is actually an $N = 2$ black-hole corresponding however to a very specific choice of the special Kähler manifold, namely $\frac{SO^*(12)}{U(6)}$ as in eq.(14), (15).

2 N=8 Supergravity and its scalar manifold $E_{7(7)}/SU(8)$

The bosonic Lagrangian of $N = 8$ supergravity contains, besides the metric 28 vector fields and 70 scalar fields spanning the $E_{7(7)}/SU(8)$ coset manifold. This lagrangian falls into the general type of lagrangians admitting electric–magnetic duality rotations considered in [22],[23],[24]. For the case where all the scalar fields of the coset manifold have been switched on the Lagrangian, according to the normalizations of [22] has the form:

$$\mathcal{L} = \sqrt{-g}\left(2\,R[g] + \frac{1}{4}\,\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\,\mathcal{F}^{\Lambda|\mu\nu}\,\mathcal{F}^{\Sigma}_{\mu\nu} + \frac{1}{4}\mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}\,\mathcal{F}^{\Lambda}_{\mu\nu}\,\mathcal{F}^{\Sigma}_{\rho\sigma}\,\epsilon^{\mu\nu\rho\sigma} + \frac{\alpha^2}{2}\,g_{ij}(\phi)\,\partial_\mu\phi^i\,\partial^\mu\phi^j\right) \tag{21}$$

where the indeces Λ, Σ enumerate the 28 vector fields, g_{ij} is the $E_{7(7)}$ invariant metric on the scalar coset manifold, α is a real number fixed by supersymmetry and the period matrix $\mathcal{N}_{\Lambda\Sigma}$ has the following general expression holding true for all symplectically embedded coset manifolds [25]:

$$\mathcal{N}_{\Lambda\Sigma} = h \cdot f^{-1} \tag{22}$$

The complex 28×28 matrices f, h are defined by the $Usp(56)$ realization $\mathbb{L}_{Usp}(\phi)$ of the coset representative which is related to its $Sp(56, \mathbb{R})$ counterpart $\mathbb{L}_{Sp}(\phi)$ through a Cayley transformation, as dispayed in the following formula [28]:

$$
\begin{aligned}
\mathbb{L}_{Usp}(\phi) &= \frac{1}{\sqrt{2}} \begin{pmatrix} f + \mathrm{i}h & \overline{f} + \mathrm{i}\overline{h} \\ f - \mathrm{i}h & \overline{f} - \mathrm{i}\overline{h} \end{pmatrix} \\
&\equiv \mathcal{C}\, \mathbb{L}_{Sp}(\phi)\, \mathcal{C}^{-1} \\
\mathbb{L}_{Sp}(\phi) &\equiv \exp\left[\phi^i\, T_i\right] = \begin{pmatrix} A(\phi) & B(\phi) \\ C(\phi) & D(\phi) \end{pmatrix} \\
\mathcal{C} &\equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathrm{i}\, \mathbb{1} \\ \mathbb{1} & -\mathrm{i}\, \mathbb{1} \end{pmatrix}
\end{aligned}
\tag{23}
$$

In eq. (23) we have implicitly utilized the solvable Lie algebra parametrization of the coset, by assuming that the matrices T_i ($i = 1, \ldots, 70$) constitute some basis of the solvable Lie algebra $Solv_7 = Solv\left(E_{7(7)}/SU(8)\right)$.

Obviously, in order to make eq.(23) explicit one has to choose a basis for the **56** representation of $E_{7(7)}$. In the sequel, according to our convenience, we utilize two different bases for a such a representation.

1. *The Dynkin basis.* In this case, hereafter referred to as $SpD(56)$, the basis vectors of the real symplectic representation are eigenstates of the Cartan generators with eigenvalue one of the 56 weight vectors ($\pm\vec{\Lambda} = \{\Lambda_1, \ldots, \Lambda_7\}$ pertaining to the representation:

$$
\begin{aligned}
(W = 1, \ldots 56) \quad : \quad |\, W\, \rangle &= \begin{cases} |\vec{\Lambda}\rangle \;\; : \;\; H_i |\, \vec{\Lambda}\, \rangle = \Lambda_i\, |\, \vec{\Lambda}\, \rangle & (\Lambda = 1, \ldots 28) \\ |\, -\vec{\Lambda}\rangle \; : \;\; H_i |\, -\vec{\Lambda}\, \rangle = -\Lambda_i\, |\, -\vec{\Lambda}\, \rangle & (\Lambda = 1, \ldots 28) \end{cases} \\
|\, V\, \rangle &= f^\Lambda\, |\vec{\Lambda}\rangle \;\oplus\; g_\Lambda\, |-\vec{\Lambda}\rangle
\end{aligned}
$$

   or in matrix notation

$$
\vec{V}_{SpD} = \begin{pmatrix} f^\Lambda \\ g_\Sigma \end{pmatrix} \tag{24}
$$

2. *The Young basis.* In this case, hereafter referred to as $UspY(56)$, the basis vectors of the complex pseudounitary representation correspond to the natural basis of the $\mathbf{28} + \overline{\mathbf{28}}$ antisymmetric representation of the maximal compact subgroup $SU(8)$. In other words, in this realization of the fundamental $E_{7(7)}$ representation a generic vector is of the following form:

$$
|V\rangle = u^{AB} \begin{array}{|c|}\hline A \\ \hline B \\ \hline \end{array} \oplus v_{AB} \begin{array}{|c|}\hline \overline{A} \\ \hline \overline{B} \\ \hline \end{array} \quad ; \quad (A, B = 1, \ldots, 8)
$$

   or in matrix notation

$$
\vec{V}_{UspY} = \begin{pmatrix} u^{AB} \\ v_{AB} \end{pmatrix} \tag{25}
$$

Although their definitions are respectively given in terms of the real and the complex case, via a Cayley transformation each of the two basis has both a real symplectic and a complex pseudounitary realization. Hence we will actually deal with four bases:

1. The $SpD(56)$–basis
2. The $Usp_D(56)$–basis
3. The $SpY(56)$–basis
4. The $UspY(56)$–basis

Each of them has distinctive advantages depending on the aspect of the theory one addresses. In particular the $UspY(56)$ basis is that originally utilized by de Wit and Nicolai in their construction of gauged $N = 8$ supergravity [26]. In the considerations of the present paper the $SpD(56)$ basis will often offer the best picture since it is that where the structure of the solvable Lie algebra is represented in the simplest way. In order to use the best features of each basis we just need to have full control on the matrix that shifts fron one to the other. We name such matrix $\mathcal{S}$ and we write:

$$\begin{pmatrix} u^{AB} \\ v_{AB} \end{pmatrix} = \mathcal{S} \begin{pmatrix} f^{\Lambda} \\ g_{\Sigma} \end{pmatrix}$$

where

$$\mathcal{S} = \begin{pmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^{\star} \end{pmatrix} \mathcal{C} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{S} & \mathrm{i}\,\mathbf{S} \\ \mathbf{S}^{\star} & -\mathrm{i}\,\mathbf{S}^{\star} \end{pmatrix}$$

the 28×28 matrix $\mathbf{S}$ being unitary

$$\mathbf{S}^{\dagger}\mathbf{S} = \mathbb{1} \tag{26}$$

The explicit form of the $U(28)$ matrix $\mathbf{S}$ is given in section 5.4. The weights of the $E_{7(7)}$ **56** representation are listed in table 2 of appendix B.

2.1 Supersymmetry transformation rules and central charges

In order to obtain the $N = 8$ BPS saturated Black Holes we cannot confine ourselves to the bosonic lagrangian, but we also need the the explicit expression for the supersymmetry transformation rules of the fermions. Since the $N = 8$ theory has no matter multiplets the fermions are just the **8** spin 3/2 gravitinos and the **56** spin 1/2 dilatinos. The two numbers **8** and **56** have been written boldfaced since they also single out the dimensions of the two irreducible $SU(8)$ representations to which the two kind of fermions are respectively assigned, namely the fundamental and the three times antisymmetric:

$$\psi_{\mu|A} \leftrightarrow \boxed{A} \equiv \mathbf{8} \quad ; \quad \chi_{ABC} \leftrightarrow \begin{array}{|c|} \hline A \\ \hline B \\ \hline C \\ \hline \end{array} \equiv \mathbf{56} \tag{27}$$

Following the conventions and formalism of [28] and [29] the relevant supersymmetry transformation rules can be written as follows:

$$\begin{aligned} \delta\psi_{A\mu} &= \nabla_\mu\epsilon_A - \frac{k}{4}\, c\, T^-_{AB|\rho\sigma}\,\gamma^{\rho\sigma}\,\gamma_\mu\,\epsilon^B + \cdots \\ \delta\chi_{ABC} &= a P_{ABCD|i}\,\partial_\mu\phi^i\,\gamma^\mu\,\epsilon^D + b\, T^-_{[AB|\rho\sigma}\,\gamma^{\rho\sigma}\epsilon_{C]} + \cdots \end{aligned} \tag{28}$$

where a, b, c are numerical coefficients fixed by superspace Bianchi identities while, by definition, $T^-_{AB|\mu\nu}$ is the antiselfdual part of the graviphoton field strength and $P_{ABCD|i}$ is the vielbein of the scalar coset manifold, completely antisymmetric in $ABCD$ and satisfying the pseudoreality condition:

$$P_{ABCD} = \frac{1}{4!}\epsilon_{ABCDEFGH}\overline{P}^{EFGH}. \tag{29}$$

What we need is the explicit expression of these objects in terms of coset representatives. For the vielbein $P_{ABCD|i}$ this is easily done. Using the $UspY(56)$ basis the left invariant 1–form has the following form:

$$\mathbb{L}(\phi)^{-1}\, d\mathbb{L}(\phi) \;=\; \begin{pmatrix} \delta^{[A}_{[C}\, Q_{D]}{}^{B]} & P^{ABEF} \\ P_{CDGH} & \delta^{[E}_{[G}\, Q^{F]}{}_{H]} \end{pmatrix} \tag{30}$$

where the 1–form $Q_D{}^B$ in the **63** adjoint representation of $SU(8)$ is the connection while the 1–form P_{CDGH} in the **70** four times antisymmetric representation of $SU(8)$ is the vielbein of the coset manifold $E_{7(7)}/SU(8)$. Later we need to express the same objects in different basis but their definition is clear from eq.(30). A little more care is needed to deal with the graviphoton field strenghts. To this effect we begin by introducing the multiplet of electric and magnetic field strengths according to the standard definitions of [22],[23] [24]:

$$\vec{V}_{\mu\nu} \equiv \begin{pmatrix} F^\Lambda_{\mu\nu} \\ G_{\Sigma|\mu\nu} \end{pmatrix} \tag{31}$$

where

$$\begin{aligned} G_{\Sigma|\mu\nu} &= -\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\,\widetilde{F}^\Sigma_{\mu\nu} - \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}\,F^\Sigma_{\mu\nu} \\ \widetilde{F}^\Sigma_{\mu\nu} &= \frac{1}{2}\,\epsilon_{\mu\nu\rho\sigma}\,F^{\Sigma|\rho\sigma} \end{aligned} \tag{32}$$

The 56–component field strenght vector $\vec{V}_{\mu\nu}$ transforms in the real symplectic representation of the U–duality group $E_{7(7)}$. We can also write a column vector containing the 28 components of the graviphoton field strenghts and their complex conjugate:

$$\vec{T}_{\mu\nu} \equiv \begin{pmatrix} T_{\mu\nu}{}^{|AB} \\ T_{\mu\nu|AB} \end{pmatrix} \qquad T^{|AB}_{\mu\nu} = \left(T_{\mu\nu|AB}\right)^\star \tag{33}$$

in which the upper and lower components transform in the canonical *Young basis* of $SU(8)$ for the $\overline{\mathbf{28}}$ and **28** representation respectively.

The relation between the graviphoton field strength vectors and the electric magnetic field strenght vectors involves the coset representative in the $SpD(56)$ representation and it is the following one:

$$\vec{T}_{\mu\nu} = -\mathcal{S}\,\mathbb{C}\,\mathbb{L}^{-1}_{SpD}(\phi)\,\vec{V}_{\mu\nu} \tag{34}$$

The matrix

$$\mathbb{C} = \begin{pmatrix} \mathbf{0} & \mathbb{1} \\ -\mathbb{1} & \mathbf{0} \end{pmatrix} \tag{35}$$

is the symplectic invariant matrix. Eq.(34) reveal that the graviphotons transform under the $SU(8)$ compensators associated with the $E_{7(7)}$ rotations. To show this let $g \in E_{7(7)}$ be an element of the U–duality group, $g(\phi)$ the action of g on the 70 scalar fields and $\mathbb{D}(g)$ be the 56 matrix representing g in the real Dynkin basis. Then by definition of coset representative we can write:

$$\mathbb{D}(g)\, \mathbb{L}_{SpD}(\phi) = \mathbb{L}_{SpD}\left(g(\phi)\right)\, W_D(g,\phi) \quad ; \quad W_D(g,\phi) \in SU(8) \subset E_{7(7)} \tag{36}$$

where $W_D(g,\phi)$ is the $SU(8)$ compensator in the Dynkin basis. If we regard the graviphoton composite field as a functional of the scalars and vector field strenghts, from eq.(36) we derive:

$$\begin{aligned} T_{\mu\nu}\left(g(\phi), \mathbb{D}(g)\vec{V}\right) &= W^*_Y(g,\phi)\, T_{\mu\nu}\left(\phi, \vec{V}\right) \\ W_Y(g,\phi) &\equiv \mathcal{S}^\star\, W_D(g,\phi)\, \mathcal{S}^T \end{aligned} \tag{37}$$

where $W_Y(g,\phi)$ is the $SU(8)$ compensator in the Young basis.

It is appropriate to express the upper and lower components of $\vec{T}$ in terms of the self–dual and antiself–dual parts of the graviphoton field strenghts, since only the latters enter (28) and therefore the equations for the BPS Black–Hole.
These components are defined as follows:

$$\begin{aligned} T^{+|AB}_{\mu\nu} &= \frac{1}{2}\left(T^{|AB}_{\mu\nu} + \frac{\mathrm{i}}{2}\,\epsilon_{\mu\nu\rho\sigma} g^{\rho\lambda} g^{\sigma\pi}\, T^{|AB}_{\lambda\pi}\right) \\ T^{-}_{AB|\mu\nu} &= \frac{1}{2}\left(T_{AB|\mu\nu} - \frac{\mathrm{i}}{2}\,\epsilon_{\mu\nu\rho\sigma} g^{\rho\lambda} g^{\sigma\pi}\, T_{AB|\lambda\pi}\right) \end{aligned} \tag{38}$$

Indeed the following equalities hold true:

$$\begin{aligned} T_{\mu\nu}{}^{|AB} &= T^{+|AB}_{\mu\nu} \\ T_{\mu\nu|AB} &= T^{-}_{\mu\nu|AB} \end{aligned} \tag{39}$$

In order to understand the above properties [28], let us first rewrite equation (34) in components:

$$\begin{aligned} T_{\mu\nu}{}^{AB} &= \overline{h}^{AB}{}_{\Sigma} F^{\Sigma}_{\mu\nu} - \overline{f}^{AB|\Lambda} G_{\Lambda|\mu\nu} \\ T^{-}_{\mu\nu|AB} &= h_{AB|\Sigma} F^{\Sigma}_{\mu\nu} - f_{AB}{}^{\Lambda} G_{\Lambda|\mu\nu} \end{aligned} \tag{40}$$

having defined the matrices h, f, $\overline{h}$, $\overline{f}$ in the following way:

$$-\,\mathcal{S}\,\mathbb{C}\,\mathbb{L}^{-1}_{SpD}(\phi) = \begin{pmatrix} \overline{h} & -\overline{f} \\ h & -f \end{pmatrix} \tag{41}$$

Next step is to express the self–dual and antiself–dual components of G_Σ (defined in the same way as for $T_{\mu\nu}$ in (38))in terms of the corresponding components of F^Σ through the period matrix

$$\begin{aligned} G^{+}_{\Sigma} &= \mathcal{N}_{\Sigma\Lambda} F^{+\Lambda} \\ G^{-}_{\Sigma} &= \overline{\mathcal{N}}_{\Sigma\Lambda} F^{-\Lambda} \end{aligned} \tag{42}$$

Projecting the two equations (40) along its self–dual and antiself–dual parts, and taking into account (42), one can deduce the following conditions:

$$\begin{array}{l} T^{-|AB}_{\mu\nu} = 0 \\ T^{+}_{\mu\nu|AB} = 0 \end{array} \tag{43}$$

which imply in turn equations (39). The symplectic vector $\vec{T}_{\mu\nu}$ of the graviphoton field strenghts may therefore be rewritten in the following form:

$$\vec{T}_{\mu\nu} \equiv \begin{pmatrix} T^{+|AB}_{\mu\nu} \\ T^{-}_{\mu\nu|AB} \end{pmatrix} \tag{44}$$

These preliminaries completed we are now ready to consider the Killing spinor equation and its general implications.

3 The Black Hole ansatz and the Killing spinor equation

The BPS saturated black holes we are interested in are classical field configurations with rotational symmetry and time translation invariance. As expected on general grounds we must allow for the presence of both electric and magnetic charges. Hence we introduce the following ansatz for the elementary bosonic fields of the theory

3.1 The Black Hole ansatz

We introduce isotropic coordinates:

$$\begin{array}{rcl} \{x^\mu\} & = & \{t, \vec{x} = x^a\} \quad ; \quad a = 1,2,3 \\ r & = & \sqrt{\vec{x} \cdot \vec{x}} \end{array} \tag{45}$$

and we parametrize the metric, the vector fields and the scalar fields as follows:

$$ds^2 = \exp\left[2U(r)\right] dt^2 - \exp\left[-2U(r)\right] d\vec{x}^2 \tag{46}$$

$$F^{-\vec{\Lambda}}_{\mu\nu} = \frac{1}{4\pi} t^{\vec{\Lambda}}(r)\, E^{-}_{\mu\nu} \tag{47}$$

$$\phi^i = \phi^i(r) \tag{48}$$

where

$$t^{\vec{\Lambda}}(r) \equiv 2\pi \left(2\, p^{\vec{\Lambda}} + \mathrm{i}\, q^{\vec{\Lambda}}(r)\right) \tag{49}$$

is a 28–component complex vector whose real part is constant, while the imaginary part is a radial fucntion to be determined. We will see in a moment the physical interpretation of $p^{\vec{\Lambda}}$ and $q^{\vec{\Lambda}}(r)$. $E^{-}_{\mu\nu}$ is the unique antiself–dual 2–form in the background of the chosen metric and it reads as follows [27], [30] :

$$E^{-} = E^{-}_{\mu\nu}\, dx^\mu \wedge dx^\nu = \mathrm{i}\frac{e^{2U(r)}}{r^3}\, dt \wedge \vec{x} \cdot d\vec{x} + \frac{1}{2}\frac{x^a}{r^3}\, dx^b \wedge dx^c \epsilon_{abc} \tag{50}$$

and it is normalized so that: $\int_{S^2_\infty} E^- = 2\,\pi$.

Combining eq.(50) with eq.(47) and (49) we conclude that

$$
\begin{aligned}
p^{\vec{\Lambda}} &= \text{magnetic charge} && (51)\\
q^{\vec{\Lambda}}(r=\infty) &= \text{electric charge} && (52)
\end{aligned}
$$

At the same time we can also identify:

$$
q^{\vec{\Lambda}}(r) \,=\, r^2\,\frac{dC^{\vec{\Lambda}}(r)}{dr}\,\exp\left[C^{\vec{\Lambda}}(r)-2U\right] \tag{53}
$$

where $\exp\left[C^{\vec{\Lambda}}(r)\right]$ is a function parametrizing the electric potential:

$$
A^{\vec{\Lambda}}_{elec} \,=\, dt\,\exp\left[C^{\vec{\Lambda}}(r)\right]. \tag{54}
$$

3.2 The Killing spinor equations

We can now analyse the Killing spinor equation combining the results (39), (42) with our ansatz (48). This allows us to rewrite (40) in the following form:

$$
\begin{aligned}
T^{+\quad AB}_{\mu\nu} &= \frac{1}{4\pi}\left(\overline{h}^{AB}{}_{\Sigma}t^{\star\Sigma} - \overline{f}^{AB|\Lambda}\mathcal{N}_{\Lambda\Sigma}t^{\star\Sigma}\right)E^+_{\mu\nu}\\
T^-_{\mu\nu|AB} &= \frac{1}{4\pi}\left(h_{AB|\Sigma}t^{\Sigma} - f_{AB}{}^{\Lambda}\overline{\mathcal{N}}_{\Lambda\Sigma}t^{\Sigma}\right)E^-_{\mu\nu}
\end{aligned} \tag{55}
$$

where $E^+_{\mu\nu} = \left(E^-_{\mu\nu}\right)^\star$.

Then we can use the general result (obtained in [27], [30]):

$$
\begin{aligned}
E^-_{\mu\nu}\,\gamma^{\mu\nu} &= 2\,\mathrm{i}\frac{e^{2U(r)}}{r^3}\,\gamma_a x^a\,\gamma_0\,\frac{1}{2}\,[\mathbf{1}+\gamma_5]\\
E^+_{\mu\nu}\,\gamma^{\mu\nu} &= -2\,\mathrm{i}\frac{e^{2U(r)}}{r^3}\,\gamma_a x^a\,\gamma_0\,\frac{1}{2}\,[\mathbf{1}-\gamma_5]
\end{aligned} \tag{56}
$$

and contracting both sides of (55) with $\gamma^{\mu\nu}$ one finally gets:

$$
\begin{aligned}
T^{+\quad AB}_{\mu\nu}\gamma^{\mu\nu} &= -\frac{\mathrm{i}}{2\pi}\frac{e^{2U(r)}}{r^3}\,\gamma_a x^a\,\gamma_0\,\left(\overline{h}^{AB}{}_{\Sigma}t^{\star\Sigma} - \overline{f}^{AB|\Lambda}\mathcal{N}_{\Lambda\Sigma}t^{\star\Sigma}\right)\frac{1}{2}\,[\mathbf{1}-\gamma_5]\\
T^-_{\mu\nu|AB}\gamma^{\mu\nu} &= \frac{\mathrm{i}}{2\pi}\frac{e^{2U(r)}}{r^3}\,\gamma_a x^a\,\gamma_0\,\left(h_{AB|\Sigma}t^{\Sigma} - f_{AB}{}^{\Lambda}\overline{\mathcal{N}}_{\Lambda\Sigma}t^{\Sigma}\right)\frac{1}{2}\,[\mathbf{1}+\gamma_5]
\end{aligned} \tag{57}
$$

At this point we specialize the supersymmetry parameter to be of the form analogue to the form utilized in [27, 30]:

$$
\epsilon_A = e^{f(r)}\xi_A \tag{58}
$$

It is useful to split the $SU(8)$ index $A = 1, \ldots, 8$ into an $SU(6)$ index $X = 1, \ldots, 6$ and an $SU(2)$ index $a = 7, 8$. Since we look for BPS states belonging to *just once shortened multiplets* (*i.e.* with $N = 2$ residual supersymmetry) we require that $\xi^X = 0$, $X = 1, \ldots, 6$ and furthermore that:

$$\gamma_0 \, \xi_a = -\mathrm{i} \, \epsilon_{ab} \, \xi^b \tag{59}$$

The vanishing of the gravitino transformation rule implies conditions on both functions $U(r)$ and $f(r)$. The equation for the latter is uninteresting since it simply fixes the form of the Killing spinor parameter. The equation for U instead is relevant since it yields the form of the black hole metric. It can be written in the following form:

$$\frac{dU}{dr} = -k \frac{e^U}{r^2} \left(h_\Sigma t^\Sigma - f^\Lambda \overline{\mathcal{N}}_{\Lambda\Sigma} t^\Sigma \right) \tag{60}$$

Furthermore the 56 differential equations from the dilatino sector can be written in the form:

$$\begin{aligned} a \, P_{ABCa|i} \frac{d\phi^i}{dr} &= \frac{b}{2\pi} \frac{e^{U(r)}}{r^2} \left(h_\Sigma t^\Sigma - f^\Lambda \overline{\mathcal{N}}_{\Lambda\Sigma} t^\Sigma \right)_{[AB} \delta^b_{C]} \epsilon_{ba} \\ a \, P^{ABCa|i} \frac{d\phi^i}{dr} &= \frac{b}{2\pi} \frac{e^{U(r)}}{r^2} \left(\overline{h}_\Sigma t^{\star\Sigma} - \overline{f}^\Lambda \mathcal{N}_{\Lambda\Sigma} t^{\star\Sigma} \right)^{[AB} \delta_b^{C]} \epsilon^{ba} \end{aligned} \tag{61}$$

Suppose now that the triplet of indices $(A,\, B,\, C)$ is of the type $(X,\, Y,\, Z)$. This corresponds to projecting eq. (61) into the representation $(\mathbf{1}, \mathbf{2}, \mathbf{20})$ of $U(1) \times SU(2) \times SU(6) \subset SU(8)$. In this case however the right hand side vanishes identically:

$$a \, P_{XYZa|i} \frac{d\phi^i}{dr} = \frac{b}{2\pi} \frac{e^{U(r)}}{r^2} \left(h_\Sigma t^\Sigma - f^\Lambda \overline{\mathcal{N}}_{\Lambda\Sigma} t^\Sigma \right)_{[XY} \delta^b_{Z]} \epsilon_{ba} \equiv 0 \tag{62}$$

so that we find that the corresponding 40 scalar fields are actually constant.

In the case where the triplet of indices $(A,\, B,\, C)$ is $(X,\, Y,\, a)$ the equations may be put in the following matrix form:

$$\begin{pmatrix} P^{XY|i} \\ P_{XY|i} \end{pmatrix} \frac{d\phi^i}{dr} = \frac{b}{3\, a\, \pi} \frac{e^{U(r)}}{r^2} \begin{pmatrix} \overline{h} & -\overline{f} \\ h & -f \end{pmatrix}_{|XY} \begin{pmatrix} \mathbf{Re}(\, t\,) \\ \mathbf{Re}(\mathcal{N} t\,) \end{pmatrix} \tag{63}$$

The above equations are obtained by projecting the terms on the left and on the right side of eq. (61) (transforming respectively in the **70** and in the **56** of $SU(8)$) on the common representation $(\mathbf{1}, \mathbf{1}, \mathbf{15}) \oplus \overline{(\mathbf{1}, \mathbf{1}, \mathbf{15})}$ of the subgroup $U(1) \times SU(2) \times SU(6) \subset SU(8)$).

Finally, when the triplet $(A,\, B,\, C)$ takes the values $(X,\, b,\, c)$, the l.h.s. of eq. (61) vanishes and we are left with the equation:

$$0 = \frac{e^{U(r)}}{r^2} \left(h_\Sigma t^\Sigma - f^\Lambda \overline{\mathcal{N}}_{\Lambda\Sigma} t^\Sigma \right)_{[Xb} \epsilon_{c]a} \tag{64}$$

This corresponds to the projection of eq. (61) into the representation $(\mathbf{1}, \mathbf{2}, \mathbf{6}) + \overline{(\mathbf{1}, \mathbf{2}, \mathbf{6})} \subset \mathbf{28} + \overline{\mathbf{28}}$.

Let us consider the basis vectors $|\vec{\Lambda}>, |-\vec{\Lambda}> \in \mathbf{56} = \mathbf{28} + \overline{\mathbf{28}}$ defined in ref. (24) and let us introduce the eigenmatrices $\mathbb{K}^{\vec{\lambda}}$ of the $SU(8)$ Cartan generators $\mathcal{H}_i$ diagonalized on the subspace of non-compact $E_{7(7)}$ generators $\mathbb{K}$ defined by the Lie algebra Cartan decomposition $E_{7(7)} = \mathbb{K} \oplus \mathbb{H}$ ($\vec{\lambda}$ being the weights of the $\mathbf{70}$ of $SU(8)$). It is convenient to use a real basis for both representations $\mathbf{56}$ and $\mathbf{70}$, namely:

$$
\begin{aligned}
|\vec{\Lambda}_x> &= \frac{|\vec{\Lambda}>+|-\vec{\Lambda}>}{2} \\
|\vec{\Lambda}_y> &= \frac{|\vec{\Lambda}>-|-\vec{\Lambda}>}{2\mathrm{i}} \\
\mathbb{K}_x^{\vec{\lambda}} &= \frac{\mathbb{K}^{\vec{\lambda}}+\mathbb{K}^{-\vec{\lambda}}}{2} \\
\mathbb{K}_y^{\vec{\lambda}} &= \frac{\mathbb{K}^{\vec{\lambda}}-\mathbb{K}^{-\vec{\lambda}}}{2\mathrm{i}}
\end{aligned} \tag{65}
$$

such that they satisfy the following relations:

$$
\begin{array}{lcrcl}
\text{projectors on irrep } \mathbf{70} & : & \left[\mathcal{H}_i\,,\,\mathbb{K}_x^{\vec{\lambda}}\right] & = & \left(\vec{\lambda},\vec{a}_i\right)\mathbb{K}_y^{\vec{\lambda}} \\
 & & \left[\mathcal{H}_i\,,\,\mathbb{K}_y^{\vec{\lambda}}\right] & = & -\left(\vec{\lambda},\vec{a}_i\right)\mathbb{K}_x^{\vec{\lambda}} \\
\text{projectors on irrep } \mathbf{28} & : & \mathcal{H}_i|\,\vec{\Lambda}_x\,\rangle & = & \left(\vec{\Lambda},\vec{a}_i\right)|\,\vec{\Lambda}_y\,\rangle \\
 & & \mathcal{H}_i|\,\vec{\Lambda}_y\,\rangle & = & -\left(\vec{\Lambda},\vec{a}_i\right)|\,\vec{\Lambda}_x\,\rangle
\end{array} \tag{66}
$$

As it will be explained in section 5.1, the seven Cartan generators $\mathcal{H}_i$ are given by appropriate linear combinations of the $E_{7(7)}$ step operators (see eq.s (152)).

Using the definitions given above, it is possible to rewrite equations (60) and (61) in an algebraically intrinsic way, which, as we will see, will prove to be useful following. This purpose can be achieved by expressing the l.h.s. and r.h.s. of eqs. (60) and (61) in terms of suitable projections on the real bases $\mathbb{K}_{x,y}^{\vec{\lambda}}$ and $|\vec{\Lambda}_{x,y}>$ respectively. In particular eqn. (62), (64) become respectively:

$$
\begin{array}{rclcrcl}
\mathrm{Tr}\left(\mathbb{K}_x^{\vec{\lambda}}\,\mathbb{L}^{-1}d\mathbb{L}\right) & = & 0 & ; & \vec{\lambda} & \in & (\mathbf{1},\mathbf{2},\mathbf{20}) \subset \mathbf{70} \\
0 & = & \langle\,\vec{\Lambda}_x = \vec{\lambda}_D|\,\mathbb{C}\,\mathbb{L}^{-1}(\phi)\,|\,\vec{t},\phi\,\rangle & ; & \vec{\Lambda} & \in & (\mathbf{1},\mathbf{2},\mathbf{6}) \subset \mathbf{28} \\
0 & = & \langle\,\vec{\Lambda}_y = \vec{\lambda}_D|\,\mathbb{C}\,\mathbb{L}^{-1}(\phi)\,|\,\vec{t},\phi\,\rangle & ; & \vec{\Lambda} & \in & (\mathbf{1},\mathbf{2},\mathbf{6}) \subset \mathbf{28}+\overline{\mathbf{28}}
\end{array} \tag{67}
$$

where we have set:

$$
|\,\vec{t},\phi\,\rangle = \begin{pmatrix} \mathbf{Re}(\,t\,) \\ \mathbf{Re}(\,\overline{\mathcal{N}}t\,) \end{pmatrix} \tag{68}
$$

The real vectors $|\vec{\Lambda}_x>$ and $|\vec{\Lambda}_y>$ in eq. (67) are related by eq. (65) to $|\vec{\Lambda}>$ and $-|\vec{\Lambda}>$, which are now restricted to the subrepresentations $(\mathbf{1},\mathbf{2},\mathbf{6})$ and $\overline{(\mathbf{1},\mathbf{2},\mathbf{6})}$ respectively. The pair of eq.s (67) can now be read as very clear statements. The first in the pair tells us that 40 out of the 70 scalar fields in the theory must be constants in the radial variable. Comparison with the results of section 6 shows that the fourty constant fields are those belonging to the solvable subalgebra $Solv_4 \subset Sol_7$ defined in eq.(14).

Hence those scalars that in an N=2 truncation belong to hypermultiplets are constant in any BPS black hole solution.

The second equation in the pair (67) can be read as a statement on the available charges. Indeed since it must be zero everywhere, the right hand side of this equation can be evaluated at infinity where the vector $|\vec{t}, \phi\rangle$ becomes the 56-component vector of electric and magnetic charges defined as:

$$\begin{pmatrix} g^{\vec{\Lambda}} \\ e_{\vec{\Sigma}} \end{pmatrix} = \begin{pmatrix} \int_{S^2_\infty} F^{\vec{\Lambda}} \\ \int_{S^2_\infty} G_{\vec{\Sigma}} \end{pmatrix} \tag{69}$$

As we will see in the following, the explicit evaluation of (70) implies that 24 combinations of the charges are zero. This, together with the fact that the last of eqn. (67) yields the vanishing of 24 more independent combinations, implies that there are only 8 surviving charges.

On the other hand, eq. (63) can be rewritten in the following form:

$$\mathrm{Tr}\left(\mathbb{K}^{\vec{\lambda}}\,\mathbb{L}^{-1}d\mathbb{L}\right) = \frac{b}{3\,a\,\pi}\frac{e^{U(r)}}{r^2}\,\langle\,\vec{\Lambda} = \vec{\lambda}|_D\,\mathbb{C}\,\mathbb{L}^{-1}(\phi)\,|\,\vec{t}, \phi\,\rangle \tag{70}$$

In equation (70) both the weights $\vec{\lambda}$ and $\vec{\Lambda}$ defining the projections in the l.h.s. and r.h.s. belong to the common representation $(\mathbf{1},\mathbf{1},\mathbf{15}) \oplus \overline{(\mathbf{1},\mathbf{1},\mathbf{15})}$ ($\lambda \in (\mathbf{1},\mathbf{1},\mathbf{15}) \oplus \overline{(\mathbf{1},\mathbf{1},\mathbf{15})} \subset \mathbf{70}$, $\left(\vec{\Lambda}\,,\,-\vec{\Lambda}\right) \in (\mathbf{1},\mathbf{1},\mathbf{15}) \oplus \overline{(\mathbf{1},\mathbf{1},\mathbf{15})} \subset \mathbf{28} + \overline{\mathbf{28}}$). Finally, in this intrinsic formalism eq. (60) takes the form:

$$\frac{dU}{dr} = 2k\frac{e^U}{r^2}\langle\vec{\Lambda}|_D\,\mathbb{C}\,\mathbb{L}^{-1}(\phi)\,|\,\vec{t}, \phi\,\rangle \quad ; \quad \vec{\Lambda} \in (\mathbf{1},\mathbf{1},\mathbf{1}) \subset \mathbf{28} \tag{71}$$

Eq. (71) implies that the projection on $|\,\vec{\Lambda}_x\,\rangle$ of the right-hand side equals $\frac{dU}{dr}$ and thus gives the differential equation for U, while the projection on $|\,\vec{\Lambda}_y\,\rangle$ equals zero, which means in turn that the central charge must be real.

In the next section, restricting our attention to a simplified case where the only non-zero scalar fields are taken in the Cartan subalgebra, we show how the above implications of the Killing spinor equation can be made explicit.

4 A simplified model: BPS black-holes reduced to the Cartan subalgebra

Just as an illustrative exercise in the present section we consider the explicit BPS black–hole solutions where the only scalar fields excited out of zero are those in the Cartan subalgebra of $E_{7(7)}$. Having set to zero all the fields except these seven, we will see that the Killing spinor equation implies that 52 of the 56 electric and magnetic charges are also zero. So by restricting our attention to the Cartan subalgebra we retrieve a BPS black hole solution that depends only on 4 charges. Furthermore in this solution 4 of the 7 Cartan fields are actually set to constants and only the remaining 3 have a non trivial radial dependence. Which is which is related to the basic solvable Lie algebra decomposition (13): the 4 Cartan fields in $Solv_4$ are constants, while the 3 Cartan fields in $Solv_3$ are radial dependent. This type of solution reproduces the so

called a-model black-holes studied in the literature, but it is not the most general. However, as we shall argue in the last section, the toy model presented here misses full generality by little. Indeed the general solution that depends on 8, rather than 4 charges, involves, besides the 3 non trivial Cartan fields other 3 nilpotent fields which correspond to axions of the compactified string theory.

Let us then begin examining this simplified case. For its study it is particularly useful to utilize the Dynkin basis since there the Cartan generators have a diagonal action on the **56** representation and correspondingly on the vector fields. Namely in the Dynkin basis the matrix $\mathcal{N}_{\Lambda\Sigma}$ is purely imaginary and diagonal once the coset representative is restricted to the Cartan subalgebra. Indeed if we name $h^i(x)$ $(i = 1, \ldots 7\ )$ the Cartan subalgebra scalar fields, we can write:

$$\mathbb{L}_{SpD}\,(h) \equiv \exp\left[\vec{h}\,\cdot\,\vec{H}\right] = \begin{pmatrix} \mathbf{A}(h) & \mathbf{0} \\ \mathbf{0} & \mathbf{D}(h) \end{pmatrix}$$

where

$$\begin{aligned} \mathbf{A}(\phi)^{\vec{\Lambda}}{}_{\vec{\Sigma}} &= \delta^{\vec{\Lambda}}{}_{\vec{\Sigma}}\,\exp\left[\vec{\Lambda}\,\cdot\,\vec{h}\right] \\ \mathbf{D}(\phi)_{\vec{\Lambda}}{}^{\vec{\Sigma}} &= \delta_{\vec{\Lambda}}{}^{\vec{\Sigma}}\,\exp\left[-\,\vec{\Lambda}\,\cdot\,\vec{h}\right] \end{aligned} \tag{72}$$

so that combining equation (22) with (23) and (72) we obtain:

$$\mathcal{N}_{\vec{\Lambda}\vec{\Sigma}} = \mathrm{i}\,\left(\mathbf{A}^{-1}\,\mathbf{D}\right)_{\vec{\Lambda}\vec{\Sigma}} = \mathrm{i}\,\delta_{\vec{\Lambda}\vec{\Sigma}}\,\exp\left[-\,2\,\vec{\Lambda}\,\cdot\,\vec{h}\right] \tag{73}$$

Hence in the Dynkin basis the lagrangian (21) reduced to the Cartan sector takes the following form:

$$\mathcal{L} = \sqrt{-g}\,\left(2\,R[g] - \frac{1}{4}\sum_{\vec{\Lambda}\in\Pi^+}\exp\left[-\,2\,\vec{\Lambda}\,\cdot\,\vec{h}\right]\,\mathcal{F}^{\vec{\Lambda}|\mu\nu}\,\mathcal{F}^{\vec{\Lambda}}_{\mu\nu} + \frac{\alpha^2}{2}\sum_{i=1}^{7}\partial_\mu h^i\,\partial^\mu h^i\right) \tag{74}$$

where by Π^+ we have denoted the set of positive weights for the fundamental representation of the U–duality group $E_{7(7)}$ and α is a real number fixed by supersymmetry already introduced in eq.(21).

Let us examine in detail the constraints imposed by (70),(67) and (71) on the 28 complex vectors t^Λ. Note that these vectors are naturally split in two subsets:

- The set: $t_z \equiv \{t^{17}, t^{18}, t^{23}, t^{24}\}$ that parametrizes 8 real charges which, through 4 suitable linearly independent combinations, transform in the representations $(\mathbf{1},\mathbf{1},\mathbf{1}) + \overline{(\mathbf{1},\mathbf{1},\mathbf{1})}$ $\oplus(\mathbf{1},\mathbf{1},\mathbf{15}) + \overline{(\mathbf{1},\mathbf{1},\mathbf{15})}$ and contribute to the 4 central charges of the theory.

- the remaining 24 complex vectors t_ℓ. Suitable linear combinations of these vectors transform in the representations $(\mathbf{1},\mathbf{1},\mathbf{1}) + \overline{(\mathbf{1},\mathbf{1},\mathbf{1})}$ $\oplus(\mathbf{1},\mathbf{2},\mathbf{6}) + \overline{(\mathbf{1},\mathbf{2},\mathbf{6})}$ and are orthogonal to the set t_z

Referring now to the present simplified model we can analyze the consequences of the projections (71) and (70). They are 16 complex conditions which split into:

1. a set of 4 equations whose coefficients depend only on t_z and give rise to 4 real conditions on the real and immaginary parts of t_z and 4 real differential equations on the Cartan fields $h_{1,2,7}$ belonging to vector multiplets, namely to the solvable Lie subalgebra $Solv_3$ (see section 6)

2. a set of 12 equations which contribute, together with the 12 conditions coming from the second of eq.s (67), to set the 24 t_ℓ to zero

In obtaining the above results, we used the fact that all the Cartan fields in $Solv_4$, (namely ($h_{3,4,5,6}$) which fall into hypermultiplets when the theory is $N = 2$ truncated) are constants, and thus can be set to zero (modulo duality rotations) as was discussed in the previous section.

After the projections have been taken into account we are left with a reduced set of non vanishing fields that includes only four vectors and three scalars, namely:

$$\begin{aligned} \text{vector fields} &= \begin{cases} F^{\Lambda^{17}}_{\mu\nu} \equiv \mathcal{F}^{17}_{\mu\nu} \\ F^{\Lambda^{18}}_{\mu\nu} \equiv \mathcal{F}^{18}_{\mu\nu} \\ F^{\Lambda^{23}}_{\mu\nu} \equiv \mathcal{F}^{23}_{\mu\nu} \\ F^{\Lambda^{24}}_{\mu\nu} \equiv \mathcal{F}^{24}_{\mu\nu} \end{cases} \\ \text{scalar fields} &= \begin{cases} h_1 \\ h_2 \\ h_7 \end{cases} \end{aligned} \tag{75}$$

In terms of these fields, using the scalar products displayed in table 3, the lagrangian has the following explicit expression:

$$\begin{aligned} \mathcal{L} &= \sqrt{-g}\Bigg\{ 2\,R[g] + \frac{\alpha^2}{2}\,\left[(\partial_\mu h_1)^2 + (\partial_\mu h_2)^2 + (\partial_\mu h_3)^2\right] \\ &\quad - \frac{1}{4}\,\exp\left[2\sqrt{\frac{2}{3}}\,h_1 - \frac{2}{\sqrt{3}}\,h_7\right]\left(\mathcal{F}^{17}_{\mu\nu}\right)^2 - \frac{1}{4}\,\exp\left[2\sqrt{\frac{2}{3}}\,h_2 - \frac{2}{\sqrt{3}}\,h_7\right]\left(\mathcal{F}^{18}_{\mu\nu}\right)^2 \\ &\quad - \frac{1}{4}\,\exp\left[2\sqrt{\frac{2}{3}}\,h_1 + \frac{2}{\sqrt{3}}\,h_7\right]\left(\mathcal{F}^{23}_{\mu\nu}\right)^2 - \frac{1}{4}\,\exp\left[2\sqrt{\frac{2}{3}}\,h_2 + \frac{2}{\sqrt{3}}\,h_7\right]\left(\mathcal{F}^{24}_{\mu\nu}\right)^2\Bigg\} \end{aligned} \tag{76}$$

Introducing an index α that takes the four values $\alpha = 17, 18, 23, 24$ for the four field strenghts, and moreover four undetermined radial functions to be fixed by the field equations:

$$q^\alpha(r) \equiv C'_\alpha e^{C_\alpha - 2U}\, r^2 \tag{77}$$

and four real constants p^α, the ansatz for the vector fields can be parametrized as follows:

$$\begin{aligned}
\mathcal{F}^\alpha_{el} &= -\frac{q^\alpha(r)\, e^{2U(r)}}{r^3}\, dt \wedge \vec{x}\cdot d\vec{x} \equiv -\frac{C'_\alpha e^{C_\alpha}}{r} dt \wedge \vec{x}\cdot d\vec{x} \\
\mathcal{F}^\alpha_{mag} &= \frac{p^\alpha}{2r^3} x^a dx^b \wedge dx^c \epsilon_{abc} \\
\mathcal{F}^\alpha &= \mathcal{F}^\alpha_{el} + \mathcal{F}^\alpha_{mag} \\
\mathcal{F}^{-\alpha}_{\mu\nu} &= \frac{1}{4\pi}\, t^\alpha E^-_{\mu\nu} \\
t^\alpha &= 2\pi\left(2p^\alpha + \mathrm{i} q^\alpha(r)\right) \equiv 2\pi(2p^\alpha + \mathrm{i}C'_\alpha e^{C_\alpha - 2U} r^2)
\end{aligned} \tag{78}$$

The physical interpretation of the above data is the following:

$$p^\alpha = \text{mag. charges} \quad ; \quad q^\alpha(\infty) = \text{elec. charges} \tag{79}$$

From the effective lagrangian of the reduced system we derive the following set of Maxwell-Einstein-dilaton field equations, where in addition to the index α enumerating the vector fields an index i taking the three values $i = 1, 2, 7$ for the corresponding three scalar fields has also been introduced:

$$\begin{aligned}
&\text{Einstein eq.} \quad : \quad -2R_{MN} = T_{MN} = \frac{1}{2}\alpha^2 \sum_i \partial_M h_i \partial_N h_i + S_{MN} \\
&\left(S_{MN} \equiv -\frac{1}{2}\sum_\alpha e^{-2\vec{\Lambda}_\alpha \cdot h} \left[\mathcal{F}^\alpha_{M\cdot}\mathcal{F}^\alpha_{N\cdot} - \frac{1}{4}\mathcal{F}^\alpha_{\cdot\cdot}\mathcal{F}^\alpha_{\cdot\cdot}\eta_{MN}\right]\right) \\
&\text{Maxwell eq.} \quad : \quad \partial_\mu \left(\sqrt{-g}\exp\left[-2\vec{\Lambda}_\alpha \cdot h\right] F^{\alpha|\mu\nu}\right) = 0 \\
&\text{Dilaton eq.s} \quad : \quad \frac{\alpha^2}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\,\partial_\nu h_i(r)\right) + \frac{1}{2}\sum_\alpha \Lambda^\alpha_i \exp[-2\,\Lambda^\alpha \cdot h]\,\mathcal{F}^\alpha_{\cdot\cdot}\mathcal{F}^\alpha_{\cdot\cdot} = 0
\end{aligned} \tag{80}$$

In eq.(80) we have denoted by dots the contraction of indices. Furthermore we have used the capital latin letters M, N for the flat Lorentz indeces obtained through multiplication by the inverse or direct vielbein according to the case. For instance:

$$\partial_M \equiv V^\mu_M\, \partial_\mu = \begin{cases} \partial_0 = e^{-U}\,\frac{\partial}{\partial t} \\ \partial_I = e^{U}\,\frac{\partial}{\partial x^i} \quad (I = 1, 2, 3) \end{cases} \tag{81}$$

Finally the components of the four relevant weights, restricted to the three relevant scalar fields are:

$$\begin{aligned}
\vec{\Lambda}_{17} &= \left(-\sqrt{\frac{2}{3}}, 0, \sqrt{\frac{1}{3}}\right) \\
\vec{\Lambda}_{18} &= \left(0, -\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right)
\end{aligned}$$

$$
\begin{aligned}
\vec{\Lambda}_{23} &= \left(-\sqrt{\frac{2}{3}}, 0, -\sqrt{\frac{1}{3}}\right) \\
\vec{\Lambda}_{24} &= \left(0, -\sqrt{\frac{2}{3}}, -\sqrt{\frac{1}{3}}\right)
\end{aligned}
\tag{82}
$$

The flat indexed stress–energy tensor T_{MN} can be evaluated by direct calculation and we easily obtain:

$$
\begin{aligned}
T_{00} &= S_{00} = -\frac{1}{4}\sum_{\alpha} \exp\left[-2\vec{\Lambda}_\alpha \cdot \vec{h} + 4U\right] \frac{1}{r^4} \left[(q^\alpha(r))^2 + (p^\alpha)^2\right] \\
T_{\ell m} &= \left(\delta_{\ell m} - 2\frac{x_\ell x_m}{r^2}\right) S_{00} + \frac{\alpha^2}{2}\frac{x_\ell x_m}{r^2}\frac{\partial \vec{h}}{\partial r} \cdot \frac{\partial \vec{h}}{\partial r}
\end{aligned}
\tag{83}
$$

The next part of the calculation involves the evaluation of the flat–indexed Ricci tensor for the metric in eq.(46). From the definitions:

$$
\begin{aligned}
0 &= dV^M - \omega^{MN} \wedge V^N \eta_{NR} \\
R^{MN} &= d\omega^{MN} - \omega^{MR} \wedge \omega^{SN} \eta_{RS} \equiv R^{MN}_{RS} V^R \wedge V^S \\
V^R &= \begin{cases} V^0 = dte^U \\ V^I = dx^i e^{-U} \end{cases}
\end{aligned}
\tag{84}
$$

we obtain the spin connection:

$$
\omega^{0I} = -\frac{x^i}{r}\, dx^i\, U' \exp[2U] \quad ; \quad \omega^{IJ} = 2\frac{x^{[i}\, dx^{j]}}{r}\, U'
\tag{85}
$$

and the Ricci tensor:

$$
\begin{aligned}
R_{00} &= -\frac{1}{2}\exp[2U]\left(U'' + \frac{2}{r}U'\right) \\
R_{ij} &= \frac{x^i\, x^j}{r^2}\exp[2U]\, (U')^2 + \delta_{ij} R_{00}
\end{aligned}
\tag{86}
$$

Correspondingly the field equations reduce to a set of first order differential equations for the eight unknown functions:

$$
U(r) \quad ; \quad h_i(r) \quad ; \quad q^\alpha(r)
\tag{87}
$$

From *Einstein equations* in (80) we get the two ordinary differential equations:

$$
\begin{aligned}
U'' + \frac{2}{r}U' &= S_{00}\, \exp[-2U] \\
(U')^2 &= \left(S_{00} - \frac{\alpha^2}{4}\sum_i (h_i')^2\right) \exp[-2U]
\end{aligned}
\tag{88}
$$

from which we can eliminate the contribution of the vector fields and obtain an equation involving only the scalar fields and the metric:

$$U'' + \frac{2}{r}U' - (U')^2 - \frac{1}{4}\sum_i (h_i')^2 = 0 \tag{89}$$

From *the dilaton equations* in (80) we get the three ordinary differential equations:

$$h_i'' + \frac{2}{r}h_i' = \frac{1}{\alpha^2}\sum_\alpha \Lambda_i^\alpha \, \exp\left[-2\Lambda^\alpha \cdot h + 2U\right] \left[(q^\alpha)^2 - (p^\alpha)^2\right] \frac{1}{r^4} \tag{90}$$

Finally *form the Maxwell equations* in (80) we obtain:

$$0 = \frac{d}{dr}\left(\exp\left[-2\Lambda_\alpha \cdot h\right] \, q^\alpha(r)\right) \tag{91}$$

4.1 The first order equations from the projection $(\mathbf{1},\mathbf{1},\mathbf{15}) \oplus (\overline{\mathbf{1}},\overline{\mathbf{1}},\overline{\mathbf{15}})$

If we reconsider the general form of eq.s (70), (71), we find that out of these 32 equations 24 are identically satisfied when the fields are restricted to be non–zero only in the chosen sector. The remaining 8 non trivial equations take the following form:

$$\begin{aligned}
0 &= c\,\frac{\mathbf{e^U}}{r^2}\left(e^{\frac{\sqrt{2}\,\mathbf{h_2}-\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{p^{18}} + e^{\frac{\sqrt{2}\,\mathbf{h_1}+\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{p^{23}} - e^{\frac{\sqrt{2}\,\mathbf{h_1}-\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{q^{17}} + e^{\frac{\sqrt{2}\,\mathbf{h_2}+\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{q^{24}}\right)\\
0 &= c\,\frac{\mathbf{e^U}}{r^2}\left(e^{\frac{\sqrt{2}\,\mathbf{h_1}-\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{p^{17}} + e^{\frac{\sqrt{2}\,\mathbf{h_2}+\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{p^{24}} - e^{\frac{\sqrt{2}\,\mathbf{h_2}-\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{q^{18}} + e^{\frac{\sqrt{2}\,\mathbf{h_1}+\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{q^{23}}\right)\\
0 &= c\,\frac{\mathbf{e^U}}{r^2}\left(e^{\frac{\sqrt{2}\,\mathbf{h_2}-\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{p^{18}} - e^{\frac{\sqrt{2}\,\mathbf{h_1}+\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{p^{23}} - e^{\frac{\sqrt{2}\,\mathbf{h_1}-\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{q^{17}} - e^{\frac{\sqrt{2}\,\mathbf{h_2}+\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{q^{24}}\right)\\
0 &= c\,\frac{\mathbf{e^U}}{r^2}\left(e^{\frac{\sqrt{2}\,\mathbf{h_1}-\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{p^{17}} - e^{\frac{\sqrt{2}\,\mathbf{h_2}+\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{p^{24}} - e^{\frac{\sqrt{2}\,\mathbf{h_2}-\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{q^{18}} - e^{\frac{\sqrt{2}\,\mathbf{h_1}+\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{q^{23}}\right)\\
\mathbf{h_7'} &= \frac{c}{2}\,\frac{\mathbf{e^U}}{r^2}\left(e^{\frac{\sqrt{2}\,\mathbf{h_2}-\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{p^{18}} - e^{\frac{\sqrt{2}\,\mathbf{h_1}+\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{p^{23}} + e^{\frac{\sqrt{2}\,\mathbf{h_1}-\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{q^{17}} + e^{\frac{\sqrt{2}\,\mathbf{h_2}+\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{q^{24}}\right)\\
(\mathbf{h_1'} - \mathbf{h_2'}) &= \frac{c}{\sqrt{2}}\,\frac{\mathbf{e^U}}{r^2}\left(e^{\frac{\sqrt{2}\,\mathbf{h_1}-\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{p^{17}} + e^{\frac{\sqrt{2}\,\mathbf{h_2}+\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{p^{24}} + e^{\frac{\sqrt{2}\,\mathbf{h_2}-\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{q^{18}} - e^{\frac{\sqrt{2}\,\mathbf{h_1}+\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{q^{23}}\right)\\
(\mathbf{h_1'} + \mathbf{h_2'}) &= \frac{c}{\sqrt{2}}\,\frac{\mathbf{e^U}}{r^2}\left(e^{\frac{\sqrt{2}\,\mathbf{h_2}-\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{p^{18}} + e^{\frac{\sqrt{2}\,\mathbf{h_1}+\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{p^{23}} + e^{\frac{\sqrt{2}\,\mathbf{h_1}-\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{q^{17}} - e^{\frac{\sqrt{2}\,\mathbf{h_2}+\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{q^{24}}\right)\\
\frac{dU}{dr} &= -\frac{k}{4\sqrt{2}}\,\frac{\mathbf{e^U}}{r^2}\left(e^{\frac{\sqrt{2}\,\mathbf{h_1}-\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{p^{17}} - e^{\frac{\sqrt{2}\,\mathbf{h_2}+\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{p^{24}} + e^{\frac{\sqrt{2}\,\mathbf{h_2}-\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{q^{18}} + e^{\frac{\sqrt{2}\,\mathbf{h_1}+\mathbf{h_7}}{\sqrt{3}}}\,\mathbf{q^{23}}\right)
\end{aligned} \tag{92}$$

where we have defined the coefficient

$$c \equiv \frac{b}{18 \times \sqrt{2}\,\pi} \tag{93}$$

$b/a, k$ being the relative coefficient between the left and right hand side of equations (60), (61) respectively, which are completely fixed by the supersymmetry transformation rules of the $N = 8$ theory (28).

From the homogeneous equations of the first order system we get:

$$\begin{aligned}
q_{17}(r)\, \exp\left[-\vec{\Lambda}_{17} \cdot \vec{h}\right] &= p_{18}\, \exp\left[-\vec{\Lambda}_{18} \cdot \vec{h}\right] \\
q_{18}(r)\, \exp\left[-\vec{\Lambda}_{18} \cdot \vec{h}\right] &= p_{17}\, \exp\left[-\vec{\Lambda}_{17} \cdot \vec{h}\right] \\
q_{23}(r)\, \exp\left[-\vec{\Lambda}_{23} \cdot \vec{h}\right] &= -p_{24}\, \exp\left[-\vec{\Lambda}_{24} \cdot \vec{h}\right] \\
q_{24}(r)\, \exp\left[-\vec{\Lambda}_{24} \cdot \vec{h}\right] &= -p_{23}\, \exp\left[-\vec{\Lambda}_{23} \cdot \vec{h}\right]
\end{aligned} \tag{94}$$

Then, from the Maxwell equations we get:

$$q_\alpha(r) = A_\alpha\, \exp\left[2\vec{\Lambda}_\alpha \cdot \vec{h}\right] \tag{95}$$

where A_α are integration constants. By substituting these into the inhomogeneous first order equations one obtains:

$$\begin{aligned}
q_{17} &= q_{24} = p_{18} = p_{23} = 0 \\
h'_1 &= -h'_2 \\
h'_7 &= 0
\end{aligned} \tag{96}$$

Introducing the field:

$$\phi = \sqrt{\frac{2}{3}} h_1 - \frac{1}{\sqrt{3}} h_7 \tag{97}$$

so that:

$$\begin{aligned}
h_1 &= \sqrt{\frac{3}{2}} \left(\phi + \frac{1}{\sqrt{3}} h_7\right) \\
h_2 &= -\sqrt{\frac{3}{2}} \left(\phi + \frac{1}{\sqrt{3}} h_7 - \log B\right) \\
\phi' &= \sqrt{\frac{2}{3}} h'_1 = -\sqrt{\frac{2}{3}} h'_2
\end{aligned} \tag{98}$$

where B is an arbitrary constant, the only independent first order equations become:

$$\phi' = \frac{c}{\sqrt{3}} \frac{\mathbf{e}^{\mathbf{U}}}{r^2} \left(p_{17} e^{\phi} + p_{24} B e^{-\phi}\right) \tag{99}$$

$$U' = -\frac{k}{2\sqrt{2}} \frac{\mathbf{e}^{\mathbf{U}}}{r^2} \left(p_{17} e^{\phi} - p_{24} B e^{-\phi}\right) \tag{100}$$

and correspondingly the second order scalar field equations become:

$$
\begin{aligned}
\phi'' + \frac{2}{r}\phi' &= \frac{1}{3\alpha^2\pi^2}\left(p_{17}^2 e^{2\phi} - p_{24}^2 B^2 e^{-2\phi}\right) && (101)\\
h_7'' + \frac{2}{r}h_7' &= 0 && (102)\\
& && (103)
\end{aligned}
$$

The system of first and second order differential equations given by eqn. (99), (100), the Einstein equations (88) and the scalar fields equations (103) can now be solved and gives:

$$
\begin{aligned}
\phi &= -\frac{1}{2}\log\left(1+\frac{b}{r}\right) + \frac{1}{2}\log\left(1+\frac{d}{r}\right) && (104)\\
U &= -\frac{1}{2}\log\left(1+\frac{b}{r}\right) - \frac{1}{2}\log\left(1+\frac{d}{r}\right) + \log A && (105)
\end{aligned}
$$

with:

$$
b = -\frac{1}{\pi\sqrt{2}}p_{17} \quad ; \quad d = -\frac{1}{\pi\sqrt{2}}B p_{24} \tag{106}
$$

fixing at the same time the coefficients (which could be alternatively fixed with supersymmetry techniques):

$$
\begin{aligned}
\alpha^2 &= \frac{4}{3}\\
c &= -\frac{\sqrt{3}}{2}\\
k &= \sqrt{2}
\end{aligned} \tag{107}
$$

This concludes our discussion of the simplified model.

We can now identify the simplified $N = 8$ model (reduced to the Cartan subalgebra) that we have studied with a class of black holes well studied in the literature. These are the black–hole generating solutions of the heterotic string compactified on a six torus. As described in [31], these heterotic black–holes can be found as solutions of the following truncated action:

$$
\begin{aligned}
S^{het} = \int d^4x \sqrt{-g}\, \Big\{ & 2\,R + 2[(\partial\phi)^2 + (\partial\sigma)^2 + (\partial\rho)^2]\\
& -\frac{1}{4}\,e^{-2\phi}\left[e^{-2(\sigma+\rho)}(F_1)^2 + e^{-2(\sigma-\rho)}(F_2)^2\right.\\
& \left. + e^{2(\sigma+\rho)}(F_1)^2 + e^{2(\sigma-\rho)}(F_2)^2\right]\Big\}
\end{aligned} \tag{108}
$$

and were studied in [32],[33],[34]. The truncated action (108) is nothing else but our truncated action (76). The translation vocabulary is given by:

$$\begin{array}{rcl} h_1 & = & \frac{\sqrt{3}}{2}\,(\sigma-\phi) \\ h_2 & = & \frac{\sqrt{3}}{2}\,(-\sigma-\phi) \\ h_7 & = & \sqrt{3}\,\rho \\ & & \end{array} \quad \begin{array}{c} ; \\ ; \\ ; \\ \end{array} \quad \begin{array}{rcl} F_{17} & = & F_4 \\ F_{18} & = & F_1 \\ F_{23} & = & F_3 \\ F_{24} & = & F_2 \end{array} \tag{109}$$

As discussed in [31] the extreme multi black–hole solutions to the truncated action (108) depend on four harmonic functions $H_i(\vec{x})$ and for a single black hole solution the four harmonic functions are simply:

$$H_i = 1 + \frac{|k_i|}{r} \tag{110}$$

which introduces four electromagnetic charges. These are the four surviving charges $p_{17}, p_{23}, q_{18}, q_{24}$ that we have found in our BPS saturated solution. It was observed in [31] that among the general extremal solutions of this model only a subclass are BPS saturated states, but in the way we have derived them, namely through the Killing spinor equation, we have automatically selected the BPS saturated ones.

To make contact with the discussion in [31] let us introduce the following four harmonic functions:

$$\begin{array}{rcl} H_{17}(r) & = & 1+\frac{|g^{17}|}{r} \\ H_{18}(r) & = & 1+\frac{|e_{18}|}{r} \end{array} \quad \begin{array}{c} ; \\ ; \end{array} \quad \begin{array}{rcl} H_{24}(r) & = & 1+\frac{|g^{24}|}{r} \\ H_{23}(r) & = & 1+\frac{|e_{23}|}{r} \end{array} \tag{111}$$

where $g^{17}, g^{24}, e_{18}, e_{23}$ are four real parameters. Translating the extremal general solution of the lagrangian (108) (see eq.s (30) of [31]) into our notations we can write it as follows:

$$\begin{array}{rcll} h_1(r) & = & -\frac{\sqrt{3}}{4} & \log\left[H_{17}/H_{23}\right] \\ h_2(r) & = & -\frac{\sqrt{3}}{4} & \log\left[H_{24}/H_{18}\right] \\ h_7(r) & = & -\frac{\sqrt{3}}{4} & \log\left[H_{18}\,H_{24}/H_{17}\,H_{23}\right] \\ U(r) & = & -\frac{1}{4} & \log\left[H_{17}\,H_{18}\,H_{23}\,H_{24}\right] \\ q^{18}(r) & = & -e_{18} & H_{18}^{-2} \\ q^{23}(r) & = & -e_{23} & H_{23}^{-2} \\ p^{24} & = & g^{24} & \\ p^{17} & = & g^{17} & \end{array} \tag{112}$$

and we see that indeed $e_{18} = -q^{18}(\infty)$, $e_{23} = -q^{23}(\infty)$ are the electric charges, while $g^{24} = p^{24}$, $g^{17} = g^{17}$ are the magnetic charges for the general extremal black–hole solution.

Comparing now eq.(109) with our previous result (98) we see that having enforced the Killing spinor equation, namely the BPS condition, we have the restrictions:

$$h_1 + h_2 = \text{const} \quad ; \quad h_7 = \text{const} \tag{113}$$

which yield:

$$H_{17}^2 = H_{18}^2 \quad ; \quad H_{23}^2 = H_{24}^2 \tag{114}$$

and hence

$$e_{18} = g^{17} \quad ; \quad e_{23} = g^{24} \tag{115}$$

Hence the BPS condition imposes that the electric charges are pairwise equal to the magnetic charges.

5 Solvable Lie algebra representation

Utilizing a well established mathematical framework [14] in all extended supergravities the scalar coset manifold U/H can be identified with the group manifold of a normed solvable Lie algebra:

$$U/H \sim \exp\left[Solv\right] \tag{116}$$

The representation of the supergravity scalar manifold $\mathcal{M}_{scalar} = U/H$ as the group manifold associated with a *normed solvable Lie algebra* introduces a one–to–one correspondence between the scalar fields ϕ^I of supergravity and the generators T_I of the solvable Lie algebra $Solv\,(U/H)$. Indeed the coset representative $\mathbb{L}(U/H)$ of the homogeneous space U/H is identified with:

$$\mathbb{L}(\phi) = \exp[\phi^I \, T_I] \tag{117}$$

where $\{T_I\}$ is a basis of $Solv\,(U/H)$.

As a consequence of this fact the tangent bundle to the scalar manifold $T\mathcal{M}_{scalar}$ is identified with the solvable Lie algebra:

$$T\mathcal{M}_{scalar} \sim Solv\,(U/H) \tag{118}$$

and any algebraic property of the solvable algebra has a corresponding physical interpretation in terms of string theory massless field modes.

Furthermore, the local differential geometry of the scalar manifold is described in terms of the solvable Lie algebra structure. Given the euclidean scalar product on $Solv$:

$$< \, , \, > \quad : \quad Solv \otimes Solv \rightarrow \mathbb{R} \tag{119}$$

$$< X, Y > \; = \; < Y, X > \tag{120}$$

the covariant derivative with respect to the Levi Civita connection is given by the Nomizu operator [15]:

$$\forall X \in Solv : \mathbb{L}_X : Solv \rightarrow Solv \tag{121}$$

$$\begin{aligned} \forall X, Y, Z \in Solv \quad &: \quad 2 < Z, \mathbb{L}_X Y > \\ &= \; < Z, [X, Y] > - < X, [Y, Z] > - < Y, [X, Z] > \end{aligned} \tag{122}$$

and the Riemann curvature 2–form is given by the commutator of two Nomizu operators:

$$< W, \{[\mathbb{L}_X, \mathbb{L}_Y] - \mathbb{L}_{[X,Y]}\} Z >= R^W{}_Z(X, Y) \tag{123}$$

In the case of maximally extended supergravities in $D = 10 - r$ dimensions the scalar manifold has a universal structure:

$$\frac{U_D}{H_D} = \frac{E_{r+1(r+1)}}{H_{r+1}} \tag{124}$$

where the Lie algebra of the U_D–group $E_{r+1(r+1)}$ is the maximally non compact real section of the exceptional E_{r+1} series of the simple complex Lie Algebras and H_{r+1} is its maximally compact subalgebra [16]. As we noted in references [12, 13], the manifolds $E_{r+1(r+1)}/H_{r+1}$ share the distinctive property of being non–compact homogeneous spaces of maximal rank $r+1$, so that the associated solvable Lie algebras, such that $E_{r+1(r+1)}/H_{r+1} = \exp\left[Solv_{(r+1)}\right]$, have the particularly simple structure:

$$Solv\ (E_{r+1}/H_{r+1}) = \mathcal{H}_{r+1} \oplus_{\alpha\in\Phi^+(E_{r+1})} \mathbb{E}^\alpha \tag{125}$$

where $\mathbb{E}^\alpha \subset E_{r+1}$ is the 1–dimensional subalgebra associated with the root α and $\Phi^+(E_{r+1})$ is the positive part of the E_{r+1}–root–system.

The generators of the solvable Lie algebra are in one to one correspondence with the scalar fields of the theory. Therefore they can be characterized as NS-NS or R-R depending on their origin in compactified string theory. To identify them algebraically it suffices to select the appropriate subgroup $T_r \subset E_{r+1}$ of the U-duality group that acts as a group of T–dualities. For instance in compactifications of TypeIIA superstrings we have $T_r = SO(r,r)$. From the algebraic point of view the generators of the solvable algebra are then of three possible types:

1. Cartan generators
2. Roots that belong to the adjoint representation of the $T_r \subset E_{r+1(r+1)}$ subalgebra (= the T–duality algebra)
3. Roots which are weights of an irreducible representation of the T–duality algebra T_r

The scalar fields associated with generators of type 1 and 2 in the above list are Neveu–Schwarz fields while the fields of type 3 are Ramond–Ramond fields.

5.1 $\mathbf{U(1) \times SU(2) \times SU(6) \subset SU(8) \subset E_{7(7)}}$

In order to make formulae (70) (67),(71) explicit and in order to derive the solvable Lie algebra decompositions we are interested in a preliminary work based on standard Lie algebra techniques.

The ingredients that we have already tacitly used in the previous sections and that are needed for a thoroughful discussion of the solavable Lie algebra splitting in (13) are:

1. The explicit listing of all the positive roots of the $E_{7(7)}$ Lie algebra
2. The explicit listing of all the weight vectors of the fundamental **56** representation of $E_{7(7)}$
3. The explicit construction of the 56×56 matrices realizing the 133 generators of $E_{7(7)}$ real Lie algebra in the fundamental representation

4. The canonical Weyl–Cartan decomposition of the $SU(8)$ maximally compact subalgebra of $E_{7(7)}$. This involves the construction of a Cartan subalgebra of A_7 type made out of $E_{7(7)}$ step operators and the construction of all A_7 step operators also in terms of suitable combinations of $E_{7(7)}$ step operators.

5. The determination of the embedding $\mathbf{SU(2)} \times \mathbf{U(6)} \subset \mathbf{SU(8)} \subset \mathbf{E_{7(7)}}$.

6. The decomposition of the maximal non compact subspace $\mathbb{K} \subset \mathbf{E_{7(7)}}$ with respect to $\mathbf{U(1)} \times \mathbf{SU(2)} \times \mathbf{SU(6)}$:

$$\mathbf{70} \to (\mathbf{1},\mathbf{1},\mathbf{15}) \oplus \overline{(\mathbf{1},\mathbf{1},\mathbf{15})} \oplus (\mathbf{1},\mathbf{2},\mathbf{20})$$

7. Using the $\mathbf{56}$ representation of $\mathbf{E_{7(7)}}$ in the **Usp**-basis, the construction of the subalgebra $\mathbf{SO^*(12)}$

The work-plan described in the above points has been completed by means of a computer programme written in MATHEMATICA [35]. In the present section we just outline the logic of our calculations and we describe the results that are summarized in various tables in appendix B. In particular we explain the method to generate the matrices of the **56** representation whose explicit form is the basic tool of our calculations but that are too large and too little instructive to display on paper.

5.2 Roots and Weights and the fundamental representation of $E_{7(7)}$

Let us begin with the construction of the fundamental representation of the U–duality group.

In [12, 13] we showed that the 63–dimensional positive part $\Phi^+(E_7)$ of the E_7 root space can be decomposed as follows:

$$\Phi^+(E_7) = \mathbb{D}_1^+ \oplus \mathbb{D}_2^+ \oplus \mathbb{D}_3^+ \oplus \mathbb{D}_4^+ \oplus \mathbb{D}_5^+ \oplus \mathbb{D}_6^+ \tag{126}$$

where $\mathbb{D}_r^+$ are the maximal abelian ideals of the nested U–duality algebras $\ldots \subset E_{r(r)} \subset E_{r+1(r+1)} \subset \ldots$ in dimension $D = 10 - r$ ($\mathbb{D}_r^+$ being the ideal of $E_{r+1(r+1)}$). The dimensions of these abelian ideals is:

$$\begin{array}{rclcrcl} \dim \mathbb{D}_1 & = & 1 & ; & \dim \mathbb{D}_2 & = & 3 \\ \dim \mathbb{D}_3 & = & 6 & ; & \dim \mathbb{D}_4 & = & 10 \\ \dim \mathbb{D}_5 & = & 16 & ; & \dim \mathbb{D}_6 & = & 27 \end{array} \tag{127}$$

The filtration (126) provides a convenient way to enumerate the 63 positive roots which in [13] were associated in one–to–one way with the massless bosonic fields of compactified string theory (for instance the TypeIIA theory). We name the roots as follows:

$$\vec{\alpha}_{i,j} \in \mathbb{D}_i \quad ; \quad \begin{cases} i = 1, \ldots, 6 \\ j = 1, \ldots, \dim \mathbb{D}_i \end{cases} \tag{128}$$

Each positive root can be decomposed along a basis of simple roots α_ℓ (i=1,. . . , 7):

$$\vec{\alpha}_{i,j} = n^\ell_{i,j}\,\alpha_\ell \quad n^\ell_{i,j} \in \mathbb{Z} \tag{129}$$

It turns out that as simple roots we can choose:

$$\begin{array}{rclcrclcrcl} \alpha_1 & = & \vec{\alpha}_{6,2} & ; & \alpha_2 & = & \vec{\alpha}_{5,2} & ; & \alpha_3 & = & \vec{\alpha}_{4,2} \\ \alpha_4 & = & \vec{\alpha}_{3,2} & ; & \alpha_5 & = & \vec{\alpha}_{2,2} & ; & \alpha_6 & = & \vec{\alpha}_{2,1} \\ \alpha_7 & = & \vec{\alpha}_{1,1} & & & & & & & & \end{array} \tag{130}$$

Having fixed this basis, each root is intrinsically identified by its Dynkin labels, namely by its integer valued components in the basis (130). The listing of all positive roots is given in table 1 were we give their name (128) according to the abelian ideal filtration, their Dynkin labels and the correspondence with massless fileds in a TypeIIA toroidal compactification.

Having identified the roots, the next step for the construction of real fundamental representation $SpD(56)$ of our U–duality Lie algebra $E_{7(7)}$ is the knowledge of the corresponding weight vectors $\vec{W}$.

A particularly relevant property of the maximally non–compact real sections of a simple complex Lie algebra is that all its irreducible representations are real. $E_{7(7)}$ is the maximally non compact real section of the complex Lie algebra E_7, hence all its irreducible representations Γ are real. This implies that if an element of the weight lattice $\vec{W} \in \Lambda_w$ is a weight of a given irreducible representation $\vec{W} \in \Gamma$ then also its negative is a weight of the same representation: $-\vec{W} \in \Gamma$. Indeed changing sign to the weights corresponds to complex conjugation.

According to standard Lie algebra lore every irreducible representation of a simple Lie algebra $\mathbb{G}$ is identified by a unique *highest* weight $\vec{W}_{max}$. Furthermore all weights can be expressed as integral non–negative linear combinations of the *simple* weights $\vec{W}_\ell$ ($\ell = 1, ..., r =$ rank($\mathbb{G}$)), whose components are named the Dynkin labels of the weight. The simple weights $\vec{W}_i$ of $\mathbb{G}$ are the generators of the dual lattice to the root lattice and are defined by the condition:

$$\frac{2(\vec{W}_i\,,\,\vec{\alpha}_j)}{(\vec{\alpha}_j\,,\,\vec{\alpha}_j)} = \delta_{ij} \tag{131}$$

In the simply laced $E_{7(7)}$ case, the previous equation simplifies as follows

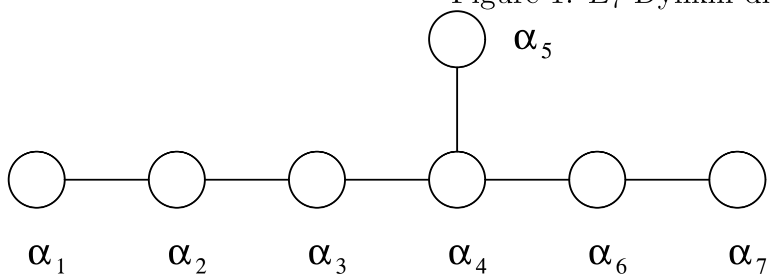
$$(\vec{W}_i\,,\,\vec{\alpha}_j) = \delta_{ij} \tag{132}$$

where $\vec{\alpha}_j$ are the the simple roots. Using eq.(130), table 1 and the Dynkin diagram of $E_{7(7)}$ (see fig.1) from eq.(132) we can easily obtain the explicit expression of the simple weights. The Dynkin labels of the highest weight of an irreducible representation Γ gives the Dynkin labels of the representation. Therefore the representation is usually denoted by $\Gamma[n_1, ..., n_r]$. All the weights $\vec{W}$ belonging to the representation Γ can be described by r integer non–negative numbers q^ℓ defined by the following equation:

$$\vec{W}_{max} - \vec{W} = \sum_{\ell=1}^{r} q^\ell \vec{\alpha}_\ell \tag{133}$$

where α_ℓ are the simple roots. According to this standard formalism the fundamental real representation $SpD(56)$ of $E_{7(7)}$ is $\Gamma[1,0,0,0,0,0,0]$ and the expression of its weights in terms of q^ℓ is given in table 2, the highest weight being $\vec{W}^{(51)}$.

Figure 1: E_7 Dynkin diagram



We can now explain the specific ordering of the weights we have adopted.

First of all we have separated the 56 weights in two groups of 28 elements so that the first group:

$$\vec{\Lambda}^{(n)} = \vec{W}^{(n)} \quad n = 1, ..., 28 \tag{134}$$

are the weights for the irreducible **28** dimensional representation of the *electric* subgroup $SL(8,\mathbb{R}) \subset E_{7(7)}$. The remaining group of 28 weight vectors are the weights for the transposed representation of the same group that we name $\overline{\mathbf{28}}$.

Secondly the 28 weights $\vec{\Lambda}$ have been arranged according to the decomposition with respect to the T$-$*duality* subalgebra $SO(6,6) \subset E_7(7)$: the first 16 correspond to R–R vectors and are the weights of the spinor representation of $SO(6,6)$ while the last 12 are associated with N–S fields and correspond to the weights of the vector representation of $SO(6,6)$.

Eq.(134) makes explicit the adopted labeling for the electric gauge fields $A_\mu^{\vec{\Lambda}}$ and their field strenghts $F_{\mu\nu}^{\vec{\Lambda}}$ adopted throughout the previous sections of the paper.

Equipped with the weight vectors we can now proceed to the explicit construction of the **SpD(56)** representation of $E_{7(7)}$. In our construction the basis vectors are the 56 weights, according to the enumeration of table 2. What we need are the 56×56 matrices associated with the 7 Cartan generators $H_{\vec{\alpha}_i}$ $(i = 1, \ldots, 7)$ and with the 126 step operators $E^{\vec{\alpha}}$ that are defined by:

$$\begin{aligned} \left[SpD_{56}\left(H_{\vec{\alpha}_i}\right)\right]_{nm} &\equiv \langle \vec{W}^{(n)} |\, H_{\vec{\alpha}_i}\, | \vec{W}^{(m)} \rangle \\ \left[SpD_{56}\left(E^{\vec{\alpha}}\right)\right]_{nm} &\equiv \langle \vec{W}^{(n)} |\, E^{\vec{\alpha}}\, | \vec{W}^{(m)} \rangle \end{aligned} \tag{135}$$

Let us begin with the Cartan generators. As a basis of the Cartan subalgebra we use the generators $H_{\vec{\alpha}_i}$ defined by the commutators:

$$\left[E^{\vec{\alpha}_i}, E^{-\vec{\alpha}_i}\right] \equiv H_{\vec{\alpha}_i} \tag{136}$$

In the $SpD(56)$ representation the corresponding matrices are diagonal and of the form:

$$\langle \vec{W}^{(p)} |\, H_{\vec{\alpha}_i}\, | \vec{W}^{(q)} \rangle = \left(\vec{W}^{(p)}, \vec{\alpha}_i \right) \delta_{pq} \quad ; \quad (p, q = 1, ..., 56) \tag{137}$$

The scalar products

$$\left(\vec{\Lambda}^{(n)} \cdot \vec{h}, -\vec{\Lambda}^{(m)} \cdot \vec{h} \right) = \left(\vec{W}^{(p)} \cdot \vec{h} \right) \quad ; \quad (n, m = 1, ..., 28\,;\, p = 1, ..., 56) \tag{138}$$

appearing in the definition 72 of the coset representative restricted to the Cartan fields, are therefore to be understood in the following way:

$$\vec{W}^{(p)} \cdot \vec{h} = \sum_{i=1}^{7} \left(\vec{W}^{(p)}, \vec{\alpha}_i\right) h^i \tag{139}$$

The explicit form of these scalar products is given in table 3

Next we construct the matrices associated with the step operators. Here the first observation is that it suffices to consider the positive roots. Indeed because of the reality of the representation, the matrix associated with the negative of a root is just the transposed of that associated with the root itself:

$$E^{-\alpha} = [E^{\alpha}]^T \leftrightarrow \langle \vec{W}^{(n)} | \, E^{-\vec{\alpha}} \, | \vec{W}^{(m)} \rangle = \langle \vec{W}^{(m)} | \, E^{\vec{\alpha}} \, | \vec{W}^{(n)} \rangle \tag{140}$$

The method we have followed to obtain the matrices for all the positive roots is that of constructing first the 56×56 matrices for the step operators $E^{\vec{\alpha}_\ell}$ $(\ell = 1, ..., 7)$ associated with the simple roots and then generating all the others through their commutators. The construction rules for the $SpD(56)$ representation of the six operators E^{α_ℓ} $(\ell \neq 5)$ are:

$$\ell \neq 5 \quad \begin{cases} \langle \vec{W}^{(n)} | \, E^{\vec{\alpha}_\ell} \, | \vec{W}^{(m)} \rangle & = & \delta_{\vec{W}^{(n)}, \vec{W}^{(m)} + \vec{\alpha}_\ell} & ; \quad n, m = 1, \ldots, 28 \\ \langle \vec{W}^{(n+28)} | \, E^{\vec{\alpha}_\ell} \, | \vec{W}^{(m+28)} \rangle & = & -\delta_{\vec{W}^{(n+28)}, \vec{W}^{(m+28)} + \vec{\alpha}_\ell} & ; \quad n, m = 1, \ldots, 28 \end{cases} \tag{141}$$

The six simple roots $\vec{\alpha}_\ell$ with $\ell \neq 5$ belong also to the the Dynkin diagram of the electric subgroup $\mathbf{SL(8,\mathbb{R})}$ (see fig.2). Thus their shift operators have a block diagonal action on the **28** and $\overline{\mathbf{28}}$ subspaces of the $SpD(56)$ representation that are irreducible under the electirc subgroup. Indeed from eq.(141) we conclude that:

$$\ell \neq 5 \quad SpD_{56}\left(E^{\vec{\alpha}_\ell}\right) = \begin{pmatrix} A[\vec{\alpha}_\ell] & \mathbf{0} \\ \mathbf{0} & -A^T[\vec{\alpha}_\ell] \end{pmatrix} \tag{142}$$

the 28×28 block $A[\vec{\alpha}_\ell]$ being defined by the first line of eq.(141).

On the contrary the operator $E^{\vec{\alpha}_5}$, corresponding to the only root of the E_7 Dynkin diagram that is not also part of the A_7 diagram is represented by a matrix whose non–vanishing 28×28 blocks are off–diagonal. We have

$$SpD_{56}\left(E^{\vec{\alpha}_5}\right) = \begin{pmatrix} \mathbf{0} & B[\vec{\alpha}_5] \\ C[\vec{\alpha}_5] & \mathbf{0} \end{pmatrix} \tag{143}$$

where both $B[\vec{\alpha}_5] = B^T[\vec{\alpha}_5]$ and $C[\vec{\alpha}_5] = C^T[\vec{\alpha}_5]$ are symmetric 28×28 matrices. More explicitly the matrix $SpD_{56}\left(E^{\vec{\alpha}_5}\right)$ is given by:

$$\begin{aligned} \langle \vec{W}^{(n)} | \, E^{\vec{\alpha}_5} \, | \vec{W}^{(m+28)} \rangle &= \langle \vec{W}^{(m)} | \, E^{\vec{\alpha}_5} \, | \vec{W}^{(n+28)} \rangle \\ \langle \vec{W}^{(n+28)} | \, E^{\vec{\alpha}_5} \, | \vec{W}^{(m)} \rangle &= \langle \vec{W}^{(m+28)} | \, E^{\vec{\alpha}_5} \, | \vec{W}^{(n)} \rangle \end{aligned} \tag{144}$$

with

$$
\begin{array}{rclrclrcl}
\langle \vec{W}^{(7)}|\, E^{\vec{\alpha}_5}\, |\vec{W}^{(44)}\rangle &=& -1 & \langle \vec{W}^{(8)}|\, E^{\vec{\alpha}_5}\, |\vec{W}^{(42)}\rangle &=& 1 & \langle \vec{W}^{(9)}|\, E^{\vec{\alpha}_5}\, |\vec{W}^{(43)}\rangle &=& -1 \\
\langle \vec{W}^{(14)}|\, E^{\vec{\alpha}_5}\, |\vec{W}^{(36)}\rangle &=& 1 & \langle \vec{W}^{(15)}|\, E^{\vec{\alpha}_5}\, |\vec{W}^{(37)}\rangle &=& -1 & \langle \vec{W}^{(16)}|\, E^{\vec{\alpha}_5}\, |\vec{W}^{(35)}\rangle &=& -1 \\
\langle \vec{W}^{(29)}|\, E^{\vec{\alpha}_5}\, |\vec{W}^{(6)}\rangle &=& -1 & \langle \vec{W}^{(34)}|\, E^{\vec{\alpha}_5}\, |\vec{W}^{(1)}\rangle &=& -1 & \langle \vec{W}^{(49)}|\, E^{\vec{\alpha}_5}\, |\vec{W}^{(28)}\rangle &=& 1 \\
\langle \vec{W}^{(50)}|\, E^{\vec{\alpha}_5}\, |\vec{W}^{(27)}\rangle &=& -1 & \langle \vec{W}^{(55)}|\, E^{\vec{\alpha}_5}\, |\vec{W}^{(22)}\rangle &=& -1 & \langle \vec{W}^{(56)}|\, E^{\vec{\alpha}_5}\, |\vec{W}^{(21)}\rangle &=& 1
\end{array}
\tag{145}
$$

In this way we have completed the construction of the $E^{\vec{\alpha}_\ell}$ operators associated with simple roots. For the matrices associated with higher roots we just proceed iteratively in the following way. As usual we organize the roots by height :

$$
\vec{\alpha} = n^\ell\, \vec{\alpha}_\ell \quad \rightarrow \quad \mathrm{ht}\, \vec{\alpha} \,=\, \sum_{\ell=1}^{7} n^\ell \tag{146}
$$

and for the roots $\alpha_i + \alpha_j$ of height $\mathrm{ht} = 2$ we set:

$$
SpD_{56}\left(E^{\alpha_i+\alpha_j}\right) \equiv \left[SpD_{56}\left(E^{\alpha_i}\right)\, ,\, SpD_{56}\left(E^{\alpha_i}\right)\right] \quad ; \quad i<j \tag{147}
$$

Next for the roots of $\mathrm{ht} = 3$ that can be written as $\alpha_i + \beta$ where α_i is simple and $\mathrm{ht}\,\beta \,=\, 2$ we write:

$$
SpD_{56}\left(E^{\alpha_i+\beta}\right) \equiv \left[SpD_{56}\left(E^{\alpha_i}\right)\, ,\, SpD_{56}\left(E^{\beta}\right)\right] \tag{148}
$$

Obtained the matrices for the roots of $\mathrm{ht} = 3$ one proceeds in a similar way for those of the next height and so on up to exhaustion of all the 63 positive roots.

This concludes our description of the algorithm by means of which our computer programme constructed all the 70 matrices spanning the solvable Lie algebra $Solv_7$ in the $SpD(56)$ representation. Taking into account property (140) once the representation of the solvable Lie algebra is given, also the remaining 63 operators corresponding to negative roots are also given.

The next point in our programme is the organization of the maximal compact subalgebra $SU(8)$ in a canonical Cartan Weyl basis. This is instrumental for a decomposition of the full algebra and of the solvable Lie algebra in particular into irreducible representations of the subgroup $U(1) \times SU(2) \times SU(6)$.

5.3 Cartan Weyl decomposition of the maximal compact subalgebra $SU(8)$

The Lie algebra $\mathbb{G}$ of $\mathbf{E_{7(7)}}$ is written, according to the Cartan decomposition, in the form:

$$
\mathbb{G} = \mathbb{H} \oplus \mathbb{K} \tag{149}
$$

where $\mathbb{H}$ denotes its maximal *compact* subalgebra (i.e. the Lie algebra of $\mathbf{SU(8)}$) and $\mathbb{K}$ its maximal *non-compact* subspace. Starting from the knowledge of the $\mathbf{E_{7(7)}}$ generators in the symplectic $SpD(56)$ representation, for brevity now denoted $H_{\alpha_i} \quad E^\alpha$, the generators in $\mathbb{H}$ and in $\mathbb{K}$ are obtained from the following identifications:

$$
\begin{aligned}
\mathbb{H} &= \left\{E^\alpha - E^{-\alpha}\right\} = \left\{E^\alpha - \left(E^\alpha\right)^T\right\} && (150)\\
\mathbb{K} &= \left\{H_{\alpha_i}; \quad E^\alpha + E^{-\alpha}\right\} = \left\{H_{\alpha_i}; \quad E^\alpha + \left(E^\alpha\right)^T\right\} && (151)
\end{aligned}
$$

In eq. (151) what is actually meant is that both $\mathbb{H}$ and $\mathbb{K}$ are the vector spaces generated by the linear combinations with *real* coefficients of the specified generators.

In order to find out the generators belonging to the subalgebra $\mathbf{U(1)} \times \mathbf{SU(2)} \times \mathbf{SU(6)}$ within $\mathbb{H}$ the generators of $\mathbb{H}$ have to be rearranged according to the canonical form of the $\mathbf{SU(8)}$ algebra. This was achieved by first fixing seven commuting matrices in $\mathbb{H}$ to be the Cartan generators of $\mathbf{SU(8)}$ and then diagonalizing with a computer programme their adjoint action over $\mathbb{H}$. Their eigenmatrices were identified with the shift operators of $\mathbf{SU(8)}$. In the sequel we will use the following notation: a will denote a generic root of $\mathbb{H}$ of the form $a = \pm\epsilon_i \pm \epsilon_j$, E^a the corresponding shift operator, $\mathcal{H}_{a_i}$ the Cartan generator associated with the simple root a_i and $B^{\alpha_{i,j}}$ the compact combination $E^{\alpha_{i,j}} - E^{-\alpha_{i,j}}$ where $\alpha_{i,j}$ is the j^{th} positive root in the i^{th} abelian ideal $\mathbb{D}_i \quad i = 1, ..., 6$ of $\mathbf{E_{7(7)}}$, according to the enumeration of table 1.

A basis $\mathcal{H}_i$ of Cartan operators was chosen as follows:

$$
\begin{aligned}
\mathcal{H}_1 &= E^{\vec{\alpha}_{2,1}} - E^{-\vec{\alpha}_{2,1}} \\
\mathcal{H}_2 &= E^{\vec{\alpha}_{2,2}} - E^{-\vec{\alpha}_{2,2}} \\
\mathcal{H}_3 &= E^{\vec{\alpha}_{4,1}} - E^{-\vec{\alpha}_{4,1}} \\
\mathcal{H}_4 &= E^{\vec{\alpha}_{4,2}} - E^{-\vec{\alpha}_{4,2}} \\
\mathcal{H}_5 &= E^{\vec{\alpha}_{6,1}} - E^{-\vec{\alpha}_{6,1}} \\
\mathcal{H}_6 &= E^{\vec{\alpha}_{6,2}} - E^{-\vec{\alpha}_{6,2}} \\
\mathcal{H}_7 &= E^{\vec{\alpha}_{6,11}} - E^{-\vec{\alpha}_{6,11}}
\end{aligned}
\tag{152}
$$

The reason for this choice is that the seven roots:

$$
\begin{array}{lcccl}
\{1,2,2,2,1,1,0\} & \leftrightarrow & \vec{\alpha}_{6,1} & = & \epsilon_1 + \epsilon_2 \\
\{1,0,0,0,0,0,0\} & \leftrightarrow & \vec{\alpha}_{6,2} & = & \epsilon_1 - \epsilon_2 \\
\{0,0,1,2,1,1,0\} & \leftrightarrow & \vec{\alpha}_{4,1} & = & \epsilon_3 + \epsilon_4 \\
\{0,0,1,0,0,0,0\} & \leftrightarrow & \vec{\alpha}_{4,1} & = & \epsilon_3 - \epsilon_4 \\
\{0,0,0,0,1,0,0\} & \leftrightarrow & \vec{\alpha}_{2,1} & = & \epsilon_5 + \epsilon_6 \\
\{0,0,0,1,0,0,0\} & \leftrightarrow & \vec{\alpha}_{2,2} & = & \epsilon_5 - \epsilon_6 \\
\{1,2,3,4,2,3,2\} & \leftrightarrow & \vec{\alpha}_{6,11} & = & \sqrt{2}\,\epsilon_7
\end{array}
\tag{153}
$$

are all orthogonal among themselves as it is evident by the last column of eq.(153) where ϵ_i denote the unit vectors in a Euclidean 7–dimensional space.

The roots $\vec{a}$ were obtained by arranging into a vector the seven eigenvalues associated with each $\mathcal{H}_i$ for a fixed eigenmatrix $E^{\vec{a}}$. Following a very well known procedure, the *positive* roots were computed as those vectors $\vec{a}$ that a positive projection along an arbitrarely fixed direction (not parallel to any of them) and among them the simple roots $\vec{a}_i$ were identified with the undecomposable ones [36]. Finally the Cartan generator corresponding to a generic root $\vec{a}$ was worked out using the following expression:

$$
\mathcal{H}_a = a^j \mathcal{H}_j
\tag{154}
$$

Once the generators of the $\mathbf{SU(8)}$ algebra were written in the canonical form, the $\mathbf{U(1)} \times$ $\mathbf{SU(2)} \times \mathbf{U(6)}$ subalgebra could be easily extracted. Choosing $\vec{a}_1$ as the root of $\mathbf{SU(2)}$ and

$\vec{a}_i \quad i = 3, ..., 7$ as the simple roots of $\mathbf{SU(6)}$, the $\mathbf{U(1)}$ generator was found to be the following combination of Cartan generators:

$$\mathcal{H}_{U(1)} = -3\mathcal{H}_{a_1} - 6\mathcal{H}_{a_2} - 5\mathcal{H}_{a_3} - 4\mathcal{H}_{a_4} - 3\mathcal{H}_{a_5} - 2\mathcal{H}_{a_6} - \mathcal{H}_{a_7} \tag{155}$$

A suitable combination of the shift operators $E^{\vec{a}}$ allowed to define the proper real compact form of the generators of $\mathbf{SU(8)}$, denoted by $X^a, \quad Y^a$. By definition, these latter fulfill the following commutation rules:

$$\begin{aligned} [\mathcal{H}_i, X^a] &= a^i Y^a && (156) \\ [\mathcal{H}_i, Y^a] &= -a^i X^a && (157) \\ [X^a, Y^a] &= a^i \mathcal{H}_i && (158) \end{aligned}$$

In tables 4, 5, 6 the explicit expressions of the generators X^a and Y^a are dislpayed as linear combinations of the step operators $B^{i,j}$. These latter are labeled according to the labeling of the $E_{7(7)}$ roots as given in table 1 where they are classified by the abelian ideal filtration. The labeling of the $SU(8)$ positive roots is the standard one according to their height. Calling $\vec{a}_1, \ldots, \vec{a}_7$ the simple roots, the full set of the 28 positive roots is the following one:

$$\vec{a}_{i,i+i,...,j-1,j} = \vec{a}_i + \vec{a}_{i+1} + \ldots + \vec{a}_{j-1} + \vec{a}_j \quad ; \quad \forall 1 \leq i < j \leq 7 \tag{159}$$

We stress that the non–compact A_7 subalgebra $SL(8, \mathbb{R})$ of $E_{7(7)}$ is regularly embedded, so that it shares the same Cartan subalgebra and its roots are vectors in the same 7–dimensional space as the roots of $E_{7(7)}$. On the other hand the compact A_7 subalgebra of $SU(8)$ is irregularly embedded and its Cartan subalgebra has actually intersection zero with the Cartan subalgebra of $E_{7(7)}$. Hence the $SU(8)$ roots are vectors in a 7–dimensional totally different from the space where the $E_{7(7)}$ roots live. Infact the Cartan generators of $SU(8)$ have been written as linear combinations of the step operators $E_{7(7)}$. The difference is emphasized in fig.2

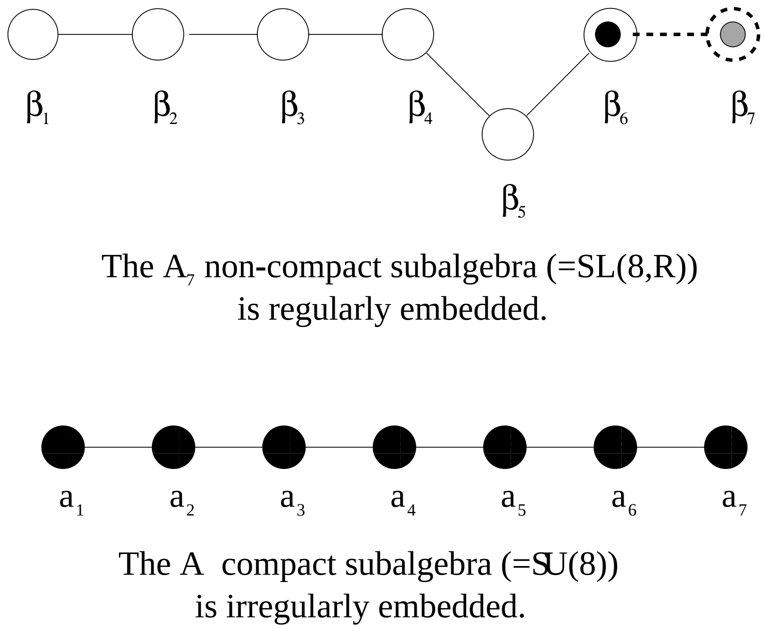
5.4 The UspY(56) basis for fundamental representation of $E_{7(7)}$

As outlined in the preceeding subsection, the generators of $\mathbf{U(1)} \times \mathbf{SU(2)} \times \mathbf{SU(6)} \subset SU(8)$ were found in terms of the matrices $B^{\alpha_{ij}} = E^{\alpha_{ij}} - E^{-\alpha_{ij}}$ belonging to the real symplectic representation $\mathbf{56}$ of $E_{7(7)}$ ($SpD(56)$). By the simultaneous diagonalization of $\mathcal{H}_{U(1)}$ and the Casimir operator of $\mathbf{SU(2)}$. it was then possible to decompose the $SpD(56)$ with respect to $\mathbf{U(1)} \times \mathbf{SU(2)} \times \mathbf{SU(6)}$ (i.e. $\mathbf{56} \rightarrow [(1,1,1) \oplus (1,1,15) \oplus (1,2,6)] \oplus \overline{[...]}$). The eigenvector basis of this decomposition provided the unitary symplectic representation $\mathbf{UspY(56)}$ in which the first diagonal block has the standard form for the Young basis:

$$\begin{aligned} T^{AB}_{CD} &= \frac{1}{2} \delta^{[A}_{[C} q_{D]}{}^{B]} && (160) \\ q_D{}^B &\in \mathbf{SU(8)} \quad A, ..., D = 1, ..., 8 && (161) \end{aligned}$$

the 8×8 matrix $q_D{}^B$ being the fundamental octet representation of the corresponding $SU(8)$ generator.

Figure 2: A_7 subalgebras of $E_{7(7)}$



Such a procedure amounts to the determination of the matrix $\mathbf{S}$ introduced in eq.(26). The explicit form of $\mathbf{S}$ is given below:

$$\mathbf{S} =$$

$$\frac{1}{\sqrt{2}}\left(\begin{array}{cccccccccccccccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{i}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0&0&0&0&\frac{1}{\sqrt{2}}&\frac{-i}{\sqrt{2}}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-i&0&0&0&0&0&0&-i&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-\frac{1}{\sqrt{3}}&0&0&0&0&0&\frac{-2\,i}{\sqrt{3}}&\frac{1}{\sqrt{3}}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-\frac{1}{\sqrt{6}}&-i\,\sqrt{\frac{3}{2}}&0&0&0&0&\frac{i}{\sqrt{6}}&\frac{1}{\sqrt{6}}&0&0&0&0\\
1&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&i&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&i&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&i&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&i&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&i&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&i&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&i&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&i\\
1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&-i&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&-i&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&-i&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&-i&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&-i&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&-i&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&-i&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&-i
\end{array}\right) \tag{162}$$

5.5 Weights of the compact subalgebra $SU(8)$

Having gained control over the embedding of the subgroup $\mathbf{U(1)} \times \mathbf{SU(2)} \times \mathbf{SU(6)}$, let us now come back to the fundamental representation of $\mathbf{E_{7(7)}}$ and consider the further decomposition

of its **28** and $\overline{\mathbf{28}}$ components, irreducible with respect to $\mathbf{SU(8)}$, when we reduce this latter to its subgroup $\mathbf{U(1)} \times \mathbf{SU(2)} \times \mathbf{SU(6)}$. In the unitary symplectic basis (either $UspD(56)$ or $UspY(56)$) the general form of an $E_{7(7)}$ Lie algebra matrix is

$$\mathcal{S} = \begin{pmatrix} T & V \\ V^* & T^* \end{pmatrix} \tag{163}$$

where T and V are 28×28 matrices respectively antihermitean and symmetric:

$$T = -T^\dagger \quad ; \quad V = V^T \tag{164}$$

The subgalgebra $SU(8)$ is represented by matrices where $V = 0$. Hence the subspaces corresponding to the first and second blocks of 28 rows are **28** and $\overline{\mathbf{28}}$ irreducible representations, respectively. Under the subgroup $\mathbf{U(1)} \times \mathbf{SU(2)} \times \mathbf{SU(6)}$ each blocks decomposes as follows:

$$\begin{aligned} \mathbf{28} &\rightarrow (\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{15}) \oplus (\mathbf{1},\mathbf{2},\mathbf{6}) \\ \overline{\mathbf{28}} &\rightarrow \overline{(\mathbf{1},\mathbf{1},\mathbf{1})} \oplus \overline{(\mathbf{1},\mathbf{1},\mathbf{15})} \oplus \overline{(\mathbf{1},\mathbf{2},\mathbf{6})} \end{aligned} \tag{165}$$

This decomposition corresponds to a rearrangement of the $\vec{\Lambda}^{(n)} = \vec{W}^{(n)}$ (and therefore $-\vec{\Lambda}^{(n)} = \vec{W}^{(n+28)}$) in a new sequence of weights $\vec{\Lambda}'^{(n)}$ ($-\vec{\Lambda}'^{(n)}$), defined in the following way:

$$\begin{aligned} \vec{\Lambda}'^{(n')} &= \vec{\Lambda}^{(n)} \\ n = 1, ..., 28 \leftrightarrow n' &= \begin{cases} [7] \leftarrow (\mathbf{1},\mathbf{1},\mathbf{1}) \\ [5,20,26,16,13,28,1,22,4,19,25,21,6,27,12] \leftarrow (\mathbf{1},\mathbf{1},\mathbf{15}) \\ [9,14,2,17,23,3,24,18,8,15,10,11] \leftarrow (\mathbf{1},\mathbf{2},\mathbf{6}) \end{cases} \end{aligned} \tag{166}$$

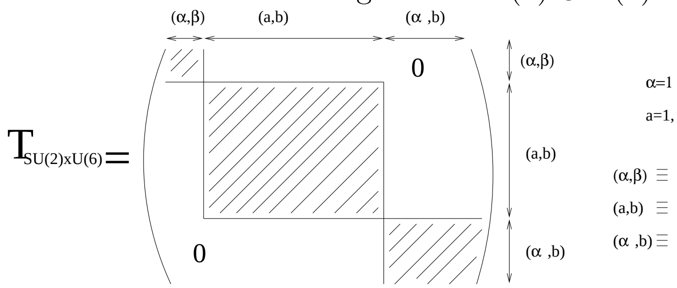
Denoting by $\vec{\Lambda}_i$ ($i = 1, ..., 7$) the simple weights of ($\mathbf{SU(8)}$), defined by an equation analogous to 132, it turns out that: $\mathbf{28} = \Gamma[0,0,0,0,0,1,0]$ and $\overline{\mathbf{28}} = \Gamma[0,1,0,0,0,0,0]$. Using the labeling (133), the weights $\vec{\Lambda}'^{(n)}$ for the **28** representation of $SU(8)$ and the weights $-\vec{\Lambda}'^{(n)}$ for the $\overline{\mathbf{28}}$ representation, ordered according to the decomposition 165, have the form listed in table 7 and 8.

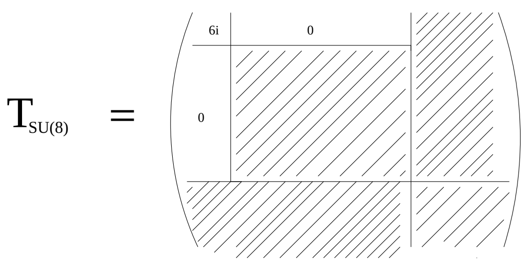
In fig.3 we show the structure of the $SU(8)$ Lie algebra elements in the $\mathbf{UspY(56)}$ basis for the fundamental representation of $E_{7(7)}$.

6 Solvable Lie algebra decompositions

In the present section, the construction of the $SO^\star(12)$ and $E_{6(4)}$ subalgebras of $E_{7(7)}$ will be discussed in detail. The starting point of this analysis is eq. (13). This equality does not uniquely define the embedding of $SO^\star(12)$ and $E_{6(4)}$ into $E_{7(7)}$. This embedding is determined by the requirement that the effective $N = 2$ theories corresponding to a truncation of the

Figure 3: $SU(2)\otimes U(6)$ and $SU(8)$ generators





$N=8$ scalar manifold to either $\mathcal{M}_3 \sim \exp Solv_3$ or $\mathcal{M}_4 \sim \exp Solv_4$, be obtained from $D=10$ type IIA theory through a compactification on a suitable Calabi–Yau manifold. This condition amounts to imposing that the fields parametrizing $Solv_3$ and $Solv_4$ should split into R–R and N–S in the following way:

$$\begin{aligned} Solv_3 &: \quad \mathbf{30} \rightarrow \mathbf{18}\,(N-S) \,+\, \mathbf{12}\,(R-R) \\ Solv_4 &: \quad \mathbf{40} \rightarrow \mathbf{20}\,(N-S) \,+\, \mathbf{20}\,(R-R) \end{aligned} \tag{167}$$

The above equations should correspond, according to a procedure defined in ([13]) and ([37]), to the decomposition of $Solv_3$ and $Solv_4$ with respect to the solvable algebra of the ST–duality group $O(6,6)\otimes SL(2,\mathbb{R})$, which is parametrized by the whole set of N–S fields in $D=4$ maximally extended supergravity. The decomposition is the following one [1]:

$$\begin{aligned} Solv\,(O(6,6)\otimes SL(2,\mathbb{R})) &= Solv\,(SU(3,3)_1)\oplus Solv\,(SU(3,3)_2)\oplus Solv\,(SL(2,\mathbb{R})) \\ Solv_3 &= Solv\,(SU(3,3)_1)\oplus \mathcal{W}_{12} \\ Solv_4 &= Solv\,(SL(2,\mathbb{R}))\oplus Solv\,(SU(3,3)_2)\oplus \mathcal{W}_{20} \end{aligned} \tag{168}$$

where $\mathcal{W}_{12}$ and $\mathcal{W}_{20}$ consist of nilpotent generators in $Solv_3$ and $Solv_4$ respectively, describing R–R fields in the **12** and **20** irreducible representations of $SU(3,3)$.

[1] For notational brevity in this section we use $Solv\,G \equiv Solv\,G/H$, H being the maximal compact subgroup of G

6.1 Structure of $SO^\star(12)$ and $E_{6(4)}$ subalgebras of $E_{7(7)}$ and some consistent $N=2$ truncations.

The subalgebras $SO^\star(12)$ and $E_{6(4)}$ in $E_{7(7)}$ were explicitly constructed starting from their maximal compact subalgebras, namely $U(6)$ and $SU(2)\otimes SU(6) \subset \mathbb{H}$, respectively. The construction of the algebra $U(1)\otimes SU(2)\otimes SU(6)\subset SU(8)$ was discussed in section 5.2. As mentioned in earlier sections, by diagonalizing the adjoint action of $U(1)$ on the 70–dimensional vector space $\mathbb{K}$, (see e.(149)) we could decompose it into irreducible representations of $U(1)\otimes SU(2)\otimes SU(6)\subset SU(8)$, namely:

$$\mathbb{K} = \mathbb{K}_{(1,1,15)} \oplus \mathbb{K}_{\overline{(1,1,15)}} \oplus \mathbb{K}_{(1,2,20)} \tag{169}$$

The algebras $SO^\star(12)$ and $E_{6(4)}$ were then constructed as follows:

$$\begin{aligned} SO^\star(12) &= \mathbb{K}_{(1,1,15)} \oplus \mathbb{K}_{\overline{(1,1,15)}} \oplus U(1) \oplus SU(6) \\ E_{6(4)} &= \mathbb{K}_{(1,2,20)} \oplus SU(2) \oplus SU(6) \end{aligned} \tag{170}$$

Unfortunately this construction does not define an embedding $Solv_3\,,\, Solv_4 \hookrightarrow Solv_7$ fulfilling the requirements (168). However this is not a serious problem. Indeed it suffices to write a new conjugate solvable Lie algebra $Solv_7^\prime = U^{-1}\, Solv_7\, U$
($U\in SU(8)/U(1)\otimes SU(2)\otimes SU(6)$) (recall that $Solv_7$ is not stable with respect to the action of $SU(8)$) such that the new embedding $Solv_3\,,\, Solv_4 \hookrightarrow Solv_7^\prime$ fulfills (168). We could easily determine such a matrix U. The unitary transformation U depends of course on the particular embedding $U(1)\otimes SU(2)\otimes SU(6)\hookrightarrow SU(8)$ chosen to define $SO^\star(12)$ and $E_{6(4)}$. Therefore, in order to achieve an interpretation of the generators of $Solv_3\,,\, Solv_4$ in terms of $N=8$ fields, the positive roots defining the two solvable algebras must be viewed as roots of $Solv_7^\prime$ whose Dynkin diagram consists of the following new simple roots:

$$\begin{aligned} \widetilde{\alpha}_1 &= \vec{\alpha}_5 \quad \widetilde{\alpha}_2 = \vec{\alpha}_{3,6} \quad \widetilde{\alpha}_3 = \vec{\alpha}_3 \\ \widetilde{\alpha}_4 &= \vec{\alpha}_2 \quad \widetilde{\alpha}_5 = \vec{\alpha}_1 \quad \widetilde{\alpha}_6 = \vec{\alpha}_{4,1} \\ \widetilde{\alpha}_7 &= -\vec{\alpha}_{6,27} \end{aligned} \tag{171}$$

Since $Solv_3$ and $Solv_4$ respectively define a special Kähler and a quaternionic manifold, it is useful to describe them in the Alekseevski's formalism [15]. The algebraic structure of $Solv_3$ and $Solv_4$ can be described in the following way:

$Solv_3$:

$$\begin{aligned} Solv_3 &= F_1 \oplus F_2 \oplus F_3 \oplus \mathbf{X} \oplus \mathbf{Y} \oplus \mathbf{Z} \\ F_i &= \{\mathrm{h}_i\,,\, \mathrm{g}_i\} \quad i=1,2,3 \\ \mathbf{X} &= \mathbf{X}^+ \oplus \mathbf{X}^- = \mathbf{X}_{NS} \oplus \mathbf{X}_{RR} \\ \mathbf{Y} &= \mathbf{Y}^+ \oplus \mathbf{Y}^- = \mathbf{Y}_{NS} \oplus \mathbf{Y}_{RR} \\ \mathbf{Z} &= \mathbf{Z}^+ \oplus \mathbf{Z}^- = \mathbf{Z}_{NS} \oplus \mathbf{Z}_{RR} \\ Solv\,(SU(3,3)_1) &= F_1 \oplus F_2 \oplus F_3 \oplus \mathbf{X}_{NS} \oplus \mathbf{Y}_{NS} \oplus \mathbf{Z}_{NS} \end{aligned}$$

$$
\begin{aligned}
Solv\left(SL(2,\mathrm{I\!R})^3\right) &= F_1 \oplus F_2 \oplus F_3 \\
\mathcal{W}_{12} &= \mathbf{X}_{RR} \oplus \mathbf{Y}_{RR} \oplus \mathbf{Z}_{RR} \\
dim\,(F_i) &= 2\,; \quad dim\left(\mathbf{X}_{NS/RR}\right) = dim\left(\mathbf{X}^{\pm}\right) = 4 \\
dim\left(\mathbf{Y}_{NS/RR}\right) &= dim\left(\mathbf{Y}^{\pm}\right) = dim\left(\mathbf{Z}_{NS/RR}\right) = dim\left(\mathbf{Z}^{\pm}\right) = 4 \\
[\mathrm{h}_i\,,\,\mathrm{g}_i] &= \mathrm{g}_i \quad i=1,2,3 \\
[F_i\,,\,F_j] &= 0 \quad i \neq j \\
\left[\mathrm{h}_3\,,\,\mathbf{Y}^{\pm}\right] &= \pm\frac{1}{2}\mathbf{Y}^{\pm} \\
\left[\mathrm{h}_3\,,\,\mathbf{X}^{\pm}\right] &= \pm\frac{1}{2}\mathbf{X}^{\pm} \\
\left[\mathrm{h}_2\,,\,\mathbf{Z}^{\pm}\right] &= \pm\frac{1}{2}\mathbf{Z}^{\pm} \\
\left[\mathrm{g}_3\,,\,\mathbf{Y}^{+}\right] &= \left[\mathrm{g}_2\,,\,\mathbf{Z}^{+}\right] = \left[\mathrm{g}_3\,,\,\mathbf{X}^{+}\right] = 0 \\
\left[\mathrm{g}_3\,,\,\mathbf{Y}^{-}\right] &= \mathbf{Y}^{+}\,;\; \left[\mathrm{g}_2\,,\,\mathbf{Z}^{-}\right] = \mathbf{Z}^{+}\,;\; \left[\mathrm{g}_3\,,\,\mathbf{X}^{-}\right] = \mathbf{X}^{+} \\
[F_1\,,\,\mathbf{X}] &= [F_2\,,\,\mathbf{Y}] = [F_3\,,\,\mathbf{Z}] = 0 \\
\left[\mathbf{X}^{-}\,,\,\mathbf{Z}^{-}\right] &= \mathbf{Y}^{-}
\end{aligned}
\tag{172}
$$

$Solv_4$:

$$
\begin{aligned}
Solv_4 &= F_0 \oplus F_1' \oplus F_2' \oplus F_2' \\
&\quad \oplus \mathbf{X}'_{NS} \oplus \mathbf{Y}'_{NS} \oplus \mathbf{Z}'_{NS} \oplus \mathcal{W}_{20} \\
Solv\,(SL(2,\mathrm{I\!R})) \oplus Solv\,(SU(3,3)_2) &= [F_0] \oplus \Big[F_1' \oplus F_2' \oplus F_2' \\
&\quad \oplus \mathbf{X}'_{NS} \oplus \Big] \\
F_0 &= \{\mathrm{h}_0\,,\,\mathrm{g}_0\} \quad [\mathrm{h}_0\,,\,\mathrm{g}_0] = \mathrm{g}_0 \\
F_i' &= \{\mathrm{h}_i'\,,\,\mathrm{g}_i'\} \quad i=1,2,3 \\
[F_0\,,\,Solv\,(SU(3,3)_2)] &= 0\,; \quad [\mathrm{h}_0\,,\,\mathcal{W}_{20}] = \frac{1}{2}\mathcal{W}_{20} \\
[\mathrm{g}_0\,,\,\mathcal{W}_{20}] &= [\mathrm{g}_0\,,\,Solv\,(SU(3,3)_2)] = 0 \\
[Solv\,(SL(2,\mathrm{I\!R})) \oplus Solv\,(SU(3,3)_2)\;,\,\mathcal{W}_{20}] &= \mathcal{W}_{20}
\end{aligned}
\tag{173}
$$

The operators $\mathrm{h}_i \quad i=1,2,3$ are the Cartan generators of $SO^\star(12)$ and g_i the corresponding axions which together with h_i complete the solvable algebra $Solv\,(SL(2,\mathrm{I\!R})^3)$ For reasons that will be apparent in the next sections we name:

$$
Solv\left(SL(2,\mathrm{I\!R})^3\right) = \text{the STU algebra} \tag{174}
$$

In order to achieve a characterization of all the $Solv\,(SO^\star(12))$ generators in terms of fields, the next step is to write down the explicit expression of the $Solv\,(SO^\star(12))$ generators in terms

of roots of $Solv_7'$, whose field interpretation can be read directly from Table 1. We have:

$$
\begin{aligned}
\mathrm{h}_1 &= \frac{1}{2}H_{\vec{\alpha}_{6,1}} \quad \mathrm{g}_1 = E^{\vec{\alpha}_{6,1}} \\
\mathrm{h}_2 &= \frac{1}{2}H_{\vec{\alpha}_{4,1}} \quad \mathrm{g}_2 = E^{\vec{\alpha}_{4,1}} \\
\mathrm{h}_3 &= \frac{1}{2}H_{\vec{\alpha}_{2,2}} \quad \mathrm{g}_3 = E^{\vec{\alpha}_{2,2}} \\
\mathbf{X}^+_{NS} &= \begin{pmatrix} E^{\vec{\alpha}_{4,3}} + E^{\vec{\alpha}_{3,4}} \\ E^{\vec{\alpha}_{3,1}} - E^{\vec{\alpha}_{4,6}} \end{pmatrix} \quad \mathbf{X}^-_{NS} = \begin{pmatrix} E^{\vec{\alpha}_{4,5}} + E^{\vec{\alpha}_{3,2}} \\ E^{\vec{\alpha}_{3,3}} - E^{\vec{\alpha}_{4,4}} \end{pmatrix} \\
\mathbf{X}^+_{RR} &= \begin{pmatrix} E^{\vec{\alpha}_{6,21}} + E^{-\vec{\alpha}_{6,17}} \\ E^{\vec{\alpha}_{5,16}} - E^{-\vec{\alpha}_{5,12}} \end{pmatrix} \quad \mathbf{X}^-_{RR} = \begin{pmatrix} E^{\vec{\alpha}_{6,20}} + E^{-\vec{\alpha}_{6,16}} \\ E^{\vec{\alpha}_{5,15}} - E^{-\vec{\alpha}_{5,11}} \end{pmatrix} \\
\mathbf{Y}^+_{NS} &= \begin{pmatrix} E^{\vec{\alpha}_{6,10}} + E^{\vec{\alpha}_{5,5}} \\ E^{\vec{\alpha}_{5,8}} - E^{\vec{\alpha}_{6,7}} \end{pmatrix} \quad \mathbf{Y}^-_{NS} = \begin{pmatrix} E^{\vec{\alpha}_{6,8}} + E^{\vec{\alpha}_{5,7}} \\ E^{\vec{\alpha}_{5,6}} - E^{\vec{\alpha}_{6,9}} \end{pmatrix} \\
\mathbf{Y}^+_{RR} &= \begin{pmatrix} E^{\vec{\alpha}_{6,24}} + E^{-\vec{\alpha}_{3,6}} \\ E^{\vec{\alpha}_{6,26}} - E^{-\vec{\alpha}_{4,9}} \end{pmatrix} \quad \mathbf{Y}^-_{RR} = \begin{pmatrix} E^{\vec{\alpha}_{6,23}} + E^{-\vec{\alpha}_{3,5}} \\ E^{\vec{\alpha}_{6,25}} - E^{-\vec{\alpha}_{4,8}} \end{pmatrix} \\
\mathbf{Z}^+_{NS} &= \begin{pmatrix} E^{\vec{\alpha}_{6,5}} + E^{\vec{\alpha}_{5,1}} \\ E^{\vec{\alpha}_{5,3}} - E^{\vec{\alpha}_{6,3}} \end{pmatrix} \quad \mathbf{Z}^-_{NS} = \begin{pmatrix} E^{\vec{\alpha}_{6,4}} - E^{\vec{\alpha}_{5,4}} \\ E^{\vec{\alpha}_{5,2}} + E^{\vec{\alpha}_{6,6}} \end{pmatrix} \\
\mathbf{Z}^+_{RR} &= \begin{pmatrix} E^{\vec{\alpha}_{6,12}} - E^{-\vec{\alpha}_{2,3}} \\ E^{\vec{\alpha}_{6,27}} + E^{-\vec{\alpha}_{1,1}} \end{pmatrix} \quad \mathbf{Z}^-_{RR} = \begin{pmatrix} E^{\vec{\alpha}_{6,22}} + E^{-\vec{\alpha}_{4,10}} \\ E^{\vec{\alpha}_{6,13}} + E^{-\vec{\alpha}_{4,7}} \end{pmatrix}
\end{aligned}
\tag{175}
$$

One can finally check that the axions associated with the STU–algebra, i.e. with the generators g_i are $B_{5,6}$, $B_{7,8}$, $g_{9,10}$. Futhermore it is worthwhile noticing that the bidimensional subalgebra F_0 of $Solv_4$ is the solvable algebra of the S–duality group $SL(2,\mathbb{R})$ of the $N = 8$ theory, and therefore is parametrized by the following fields:

$$
\begin{aligned}
\phi\,(dilaton) &\leftrightarrow \mathrm{h}_0 \\
B_{\mu\nu} &\leftrightarrow \mathrm{g}_0
\end{aligned}
\tag{176}
$$

Some consistent $N = 2$ truncations of the $N = 8$ theory can be described in terms of their scalar content in the following way:

$$
\begin{aligned}
\mathcal{M}_{N=8} &\sim Solv_7' \rightarrow \mathcal{M}_{N=2} \equiv \mathcal{M}_{vec} \otimes \mathcal{M}_{quat} \\
\mathcal{M}_{vec} &\sim Solv_3 \qquad \mathcal{M}_{quat} \sim \mathbb{1} \\
\mathcal{M}_{vec} &\sim Solv\,(SU(3,3)_1) \quad \mathcal{M}_{quat} \sim Solv\,(SU(2,1)) \\
\mathcal{M}_{vec} &\sim Solv\left(Sl(2,\mathbb{R})^3\right) \quad \mathcal{M}_{quat} \sim Solv\,(SO(4,6)) \\
\mathcal{M}_{vec} &\sim \mathbb{1} \qquad \mathcal{M}_{quat} \sim Solv\left(E_{6(4)}\right)
\end{aligned}
\tag{177}
$$

7 The general solution and conclusions

In this paper we have considered two separate but closely related issues:

1. The $N=2$ decomposition of the $N=8$ solvable Lie algebra $Solv_7 \equiv Solv\left(E_{7(7)}/SU(8)\right)$
2. The system of first and second order equations characterizing BPS black–holes in the $N=8$ theory

With respect to issue (1) our treatment has been exhaustive and we have shown how the decomposition (13),(14) corresponds to the splitting of the $N=8$ scalar fields into vector multiplet scalars and hypermultiplet scalars. We have also shown how the alekseevskian analysis of the decomposed solvable Lie algebra $Solv_7$ is the key to determine the consistent $N=2$ truncations of the $N=8$ theory at the interaction level. In addition the algebraic results on the embedding of the $U(1)\times SU(2)\times SU(6)$ Lie algebra into $E_{7(7)}$ and the solvable counterparts of this embedding are instrumental for the completion of the programme already outlined in our previous paper [13], namely the gauging of the maximal gaugeable abelian ideal $\mathcal{G}_{abel} \subset Solv_7$ which turns out to be of dimension 7. This gauging is postponed to a future publication, but the algebraic results presented in this paper are an essential step forward in this direction.

With respect to issue (2) we made a general group–theoretical analysis of the Killing vector equations and we proved that the hypermultiplet scalars corresponding to the solvable Lie subalgebra $Solv_4 \subset Solv_7$ are constant in the most general solution. Next we analysed a simplified model where the only non–zero fields are those in the Cartan subalgebra $H \subset Solv_7$ and we showed how the algebraically decomposed Killing spinor equations work in an explicit way. In particular by means of this construction we retrieved the $N=8$ embedding of the a–model black-hole solutions known in the literature [2]. It remains to be seen how general the presented solutions are, modulo U–duality rotations. That they are not fully general is evident from the fact that by restricting the non–zero fields to be in the Cartan subalgebra we obtain constraints on the electric and magnetic charges such that the solution is parametrized by only four charges: two electric (q_{18}, q_{23}) and two magnetic p_{17}, p_{24}. We are therefore lead to consider the question

How many more scalar fields besides those associated with the Cartan subalgebra have to be set non zero in order to generate the most general solution modulo U–duality rotations?

An answer can be given in terms of solvable Lie algebra once again. The argument is the following.

Let

$$\vec{Q} \equiv \begin{pmatrix} g^{\vec{\Lambda}} \\ e_{\vec{\Sigma}} \end{pmatrix} \tag{178}$$

be the vector of electric and magnetic charges (see eq.(69)) that transforms in the **56** dimensional real representation of the U duality group $E_{7(7)}$. Through the Cayley matrix we can convert it to the **Usp(56)** basis namely to:

$$\begin{pmatrix} t^{\vec{\Lambda}_1} = g^{\vec{\Lambda}_1} + \mathrm{i}\, e_{\vec{\Lambda}_1} \\ \bar{t}_{\vec{\Lambda}_1} = g^{\vec{\Lambda}_1} - \mathrm{i}\, e_{\vec{\Lambda}_1} \end{pmatrix} \tag{179}$$

Acting on $\vec{Q}$ by means of suitable $Solv\left(E_{7(7)}\right)$ transformations, we can reduce it to the following *normal* form:

$$\vec{Q} \rightarrow \vec{Q}^N \equiv \begin{pmatrix} t^0_{(1,1,1)} \\ t^1_{(1,1,15)} \\ t^2_{(1,1,15)} \\ t^3_{(1,1,15)} \\ 0 \\ \dots \\ 0 \\ \\ \bar{t}^0_{(1,1,1)} \\ \bar{t}^1_{(1,1,15)} \\ \bar{t}^2_{(1,1,15)} \\ \bar{t}^3_{(1,1,15)} \\ 0 \\ \dots \\ 0 \end{pmatrix} \tag{180}$$

Consequently also the central charge $\vec{Z} \equiv \left(Z^{AB}\,,\,Z_{CD}\right)$, which depends on $\vec{Q}$ through the coset representative in a symplectic–invariant way, will be brought to the *normal* form

$$\vec{Z} \rightarrow \vec{Z}^N \equiv \begin{pmatrix} z^0_{(1,1,1)} \\ z^1_{(1,1,15)} \\ z^2_{(1,1,15)} \\ z^3_{(1,1,15)} \\ 0 \\ \dots \\ 0 \\ \\ \bar{z}^0_{(1,1,1)} \\ \bar{z}^1_{(1,1,15)} \\ \bar{z}^2_{(1,1,15)} \\ \bar{z}^3_{(1,1,15)} \\ 0 \\ \dots \\ 0 \end{pmatrix} \tag{181}$$

through a suitable $SU(8)$ transformation. It was shown in [21], [38] that $\vec{Q}^N$ is invariant with respect to the action of an $O(4,4)$ subgroup of $E_{7(7)}$ and its *normalizer* is an $SL(2,\mathbb{R})^3 \subset E_{7(7)}$ commuting with it. Indeed it turns out that the eight real parameters in $\vec{Q}^N$ are singlets with respect to $O(4,4)$ and in a $(\mathbf{2},\mathbf{2},\mathbf{2})$ irreducible representation of $SL(2,\mathbb{R})^3$ as it is shown in the following decomposition of the $\mathbf{56}$ with respect to $O(4,4) \otimes SL(2,\mathbb{R})^3$:

$$\mathbf{56} \rightarrow (\mathbf{8_v},\mathbf{2},\mathbf{1},\mathbf{1}) \,\oplus\, (\mathbf{8_s},\mathbf{1},\mathbf{2},\mathbf{1}) \,\oplus\, (\mathbf{8_{s'}},\mathbf{1},\mathbf{1},\mathbf{2}) \,\oplus\, (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{2}) \tag{182}$$

The corresponding subgroup of $SU(8)$ leaving $\vec{Z}^N$ invariant is therefore $SU(2)^4$ which is the maximal compact subgroup of $O(4,4)$.

Note that $SL(2,\mathbb{R})^3$ contains a $U(1)^3$ which is in $SU(8)$ and which can be further used to classify the general normal frame black-holes by five real parameters, namely four complex numbers with the same phase. This corresponds to write the 56 dimensional generic vector in terms of the five normal frame parameters plus 51 "angles" which parametrize the 51 dimensional compact space $\frac{SU(8)}{SU(2)^4}$, where $SU(2)^4$ is the maximal compact subgroup of the stability group $O(4,4)$ [39].

Consider now the scalar "geodesic" potential (see eq.(34)):

$$\begin{aligned} V(\phi) &\equiv \overline{Z}^{AB}(\phi)\, Z_{AB}(\phi) \\ &= \vec{Q}^T \left[\mathbb{L}^{-1}(\phi)\right]^T \mathbb{L}^{-1}(\phi)\, \vec{Q} \end{aligned} \tag{183}$$

whose minimization determines the fixed values of the scalar fields at the horizon of the black–hole. Because of its invariance properties the scalar potential $V(\phi)$ depends on $\vec{Z}$ and therefore on $\vec{Q}$ only through their normal forms. Since the fixed scalars at the horizon of the Black–Hole are obtained minimizing $V(\phi)$, it can be inferred that the most general solution of this kind will depend (modulo duality transformations) only on those scalar fields associated with the *normalizer* of the normal form $\vec{Q}^N$. Indeed the dependence of $V(\phi)$ on a scalar field is achieved by acting on $\vec{Q}$ in the expression of $V(\phi)$ by means of the transformations in $Solv_7$ associated with that field. Since at any point of the scalar manifold $V(\phi)$ can be made to depend only on $\vec{Q}^N$, its minimum will be defined only by those scalars that correspond to trasformations acting on the non–vanishing components of the normal form (*normalizer* of $\vec{Q}^N$). Indeed all the other isometries were used to rotate $\vec{Q}$ to the normal form $\vec{Q}^N$. Among those scalars which are not determined by the fixed point conditions there are the *flat direction fields* namely those on which the scalar potential does not depend at all:

$$\text{flat direction field } q_f \quad \leftrightarrow \quad \frac{\partial}{\partial q_f} V(\phi) = 0 \tag{184}$$

Some of these fields parametrize $Solv\,(O(4,4))$ since they are associated with isometries leaving $\vec{Q}^N$ invariant, and the remaining ones are obtained from the latter by means of duality transformations. In order to identify the scalars which are *flat* directions of $V(\phi)$, let us consider the way in which $Solv\,(O(4,4))$ is embedded into $Solv_7$, referring to the description of $Solv_4$ given in eqs. (173):

$$\begin{aligned} Solv\,(O(4,4)) &\subset Solv_4 \\ Solv\,(O(4,4)) &= F_0 \oplus F_1' \oplus F_2' \oplus F_3' \oplus \mathcal{W}_8 \end{aligned} \tag{185}$$

where the R–R part $\mathcal{W}_8$ of $Solv\,(O(4,4))$ is the quaternionic image of $F_0 \oplus F_1' \oplus F_2' \oplus F_3'$ in $\mathcal{W}_{20}$. Therefore $Solv\,(O(4,4))$ is parametrized by the 4 *hypermultiplets* containing the Cartan fields of $Solv\left(E_{6(4)}\right)$. One finds that the other flat directions are all the remaining parameters of $Solv_4$, that is all the hyperscalars.

Alternatively we can observe that since the hypermultiplet scalars are flat directions of the potential, then we can use the solvable Lie algebra $Solv_4$ to set them to zero at the horizon. Since we know from the Killing spinor equations that these 40 scalars are constants it follows that we can safely set them to zero and forget about their existence (modulo U–duality transformations). Hence the non zero scalars required for a general solution have to be looked for among the vector multiplet scalars that is in the solvable Lie algebra $Solv_3$. In other words the most general $N = 8$ black–hole (up to U–duality rotations) is given by the most general $N = 2$ black–hole based on the 15–dimensional special Kähler manifold:

$$\mathcal{SK}_{15} \equiv \exp\left[Solv_3\right] = \frac{SO^\star(12)}{U(1) \times SU(6)} \tag{186}$$

Having determined the little group of the normal form enables us to decide which among the above 30 scalars have to be kept alive in order to generate the most general BPS black–hole solution (modulo U–duality).

We argue as follows. The *normalizer* of the normal form is contained in the largest subgroup of $E_{7(7)}$ commuting with $O(4,4)$. Indeed, a necessary condition for a group G^N to be the *normalizer* of $\vec{Q}^N$ is to commute with the *little group* $G^L = O(4,4)$ of $\vec{Q}^N$:

$$\begin{aligned} \vec{Q}'^N &= G^N \cdot \vec{Q}^N \quad \vec{Q}^N = G^L \cdot \vec{Q}^N \\ \vec{Q}'^N &= G^L \cdot \vec{Q}'^N \Rightarrow \left[G^N \, , \, G^L\right] = 0 \end{aligned} \tag{187}$$

As previously mentioned, it was proven that $G^N = SL(2,\mathrm{I\!R})^3 \subset SO^\star(12)$ whose solvable algebra is defined by the last of eqs. (172). Moreover G^N coincides with the largest subgroup of $Solv_7$ commuting with G^L.

The duality transformations associated with the $SL(2,\mathrm{I\!R})^3$ isometries act only on the eight non vanishing components of $\vec{Q}^N$ and therefore belong to $\mathbf{Sp(8)}$.

In conclusion the most general $N = 8$ black–hole solution is described by the 6 scalars parametrizing $Solv\,(SL(2,\mathrm{I\!R})^3)$, which are the only ones involved in the fixed point conditions at the horizon.

Another way of seeing this is to notice that all the other 64 scalars are either the 16 parameters of $Solv\,(O(4,4))$ which are flat directions of $V(\phi)$, or coefficients of the $48 = 56-8$ transformations needed to rotate $\vec{Q}$ into $\vec{Q}^N$ that is to set 48 components of $\vec{Q}$ to zero as shown in eq. (180).

Let us then reduce our attention to the Cartan vector multiplet sector, namely to the 6 vectors corresponding to the solvable Lie algebra $Solv\,(SL(2,\mathrm{I\!R}))$.

7.1 $SL(2,\mathrm{I\!R})^3$ and the fixed scalars at the horizon

In this paper we have elaborated the group–theoretical rules of this game and in section 4 we have worked out the simplified example where the only non–zero fields are in the Cartan subalgebra. From the viewpoint of string toroidal compactifications this means that we have just introduced the dilaton and the 6 radii R_i of the torus T^6. An item that so far was clearly missing are the 3 commuting axions $B_{5,6}$, $B_{7,8}$ and $g_{9,10}$. As already pointed out, by looking

at table 1 we realize that they correspond to the roots $\alpha_{6,1}, \alpha_{4,1}, \alpha_{2,2}$. So, as it is evident from eq.(175) the nilpotent generators associated with these fields are the g_1, g_2, g_3 partners of the Cartan generators h_1, h_2, h_3 completing the three 2–dimensional *key algebras* F_1, F_2, F_3 in the Alekseveeskian decomposition of the Kähler algebra $Sol_3 = Solv\,(SO^\star(12))$ (see eq.(172)):

$$Solv_3 = F_1 \oplus F_2 \oplus F_3 \oplus \mathbf{X} \oplus \mathbf{Y} \oplus \mathbf{Z} \tag{188}$$

This triplet of key algebras is nothing else but the Solvable Lie algebra of $[SL(2,\mathbb{R})/U(1)]^3$ defined above as the normalizer of the little group of the normal form $\vec{Q}^N$:

$$F_1 \oplus F_2 \oplus F_3 = Solv\,(SL(2,\mathbb{R}) \otimes SL(2,\mathbb{R}) \otimes SL(2,\mathbb{R})) \tag{189}$$

The above considerations have reduced the quest for the most general $N = 8$ black–hole to the solution of the model containing only the 6 scalar fields associated with the triplet of key algebras (189). This model is nothing else but the model of STU N=2 black-holes studied in [20]. Hence we can utilize the results of that paper and insert them in the general set up we have derived. In particular we can utilize the determination of the fixed values of the scalars at the horizon in terms of the charges given in [20]. To make a complete connection between the results of that paper and our framework we just need to derive the relation between the fields of the solvable Lie algebra parametrization and the standard S, T, U complex fields utilized as coordinates of the special Kähler manifold:

$$\mathcal{ST}[2,2] \equiv \frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,2)}{SO(2)\times SO(2)} \tag{190}$$

To this effect we consider the embedding of the Lie algebra $SL(2,\mathbb{R})_1 \times SL(2,\mathbb{R})_2 \times SL(2,\mathbb{R})_3$ into $Sp(8,\mathbb{R})$ such that the fundamental $\mathbf{8}$–dimensional representation of $Sp(8,\mathbb{R})$ is irreducible under the three subgroups and is

$$\mathbf{8} = (\mathbf{2},\mathbf{2},\mathbf{2}) \tag{191}$$

The motivation of this embedding is that the $SO(4,4)$ singlets in the decomposition (182) transform under $SL(2,\mathbb{R})^3$ as the representation mentioned in eq.(191). Therefore the requested embedding corresponds to the action of the *key algebras* $F_1 \oplus F_2 \oplus F_3$ on the non vanishing components of the charge vector in its normal form. We obtain the desired result from the standard embedding of $SL(2,\mathbb{R}) \times SO(2,n)$ in $Sp\,(2\times(2+n),\mathbb{R})$:

$$\begin{aligned} \mathbf{A} \in SO(2,n) &\hookrightarrow \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & -\mathbf{A}^T \end{pmatrix} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) &\hookrightarrow \begin{pmatrix} a\,\mathbb{1} & b\,\eta \\ c\,\eta & d\,\mathbb{1} \end{pmatrix} \end{aligned} \tag{192}$$

used to derive the Calabi Vesentini parametrization of the Special Kähler manifold:

$$\mathcal{ST}[2,n] \equiv \frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2)\times SO(n)} \tag{193}$$

It suffices to set $n = 2$ and to use the accidental isomorphism:

$$SO(2,2) \sim SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \tag{194}$$

Correspondingly we can write an explicit realization of the $SL(2,\mathbb{R})^3$ Lie algebra:

$$\begin{aligned} \left[L_0^{(i)}\,,\,L_\pm^{(i)}\right] &= \pm\,L_\pm^{(i)} \quad i=1,2,3 \\ \left[L_+^{(i)}\,,\,L_-^{(i)}\right] &= 2\,L_0^{(i)} \quad i=1,2,3 \\ \left[L_A^{(i)}\,,\,L_B^{(j)}\right] &= 0 \quad i\neq j \end{aligned} \tag{195}$$

by means of 8×8 symplectic matrices satisfying:

$$\left[L_A^{(i)}\right]^T \mathbb{C} + \mathbb{C}\,L_A^{(i)} = 0 \tag{196}$$

where

$$\mathbb{C} = \begin{pmatrix} \mathbf{0}_{4\times4} & \mathbb{1}_{4\times4} \\ -\mathbb{1}_{4\times4} & \mathbf{0}_{4\times4} \end{pmatrix} \tag{197}$$

Given this structure of the algebra, we can easily construct the coset representatives by writing:

$$\begin{aligned} \mathbb{L}^{(i)}\,(h_i,a_i) &\equiv \exp[2\,h_i\,L_0^{(i)}]\,\exp[a_i\,e^{-h_i}\,L_+^{(i)}] \\ &= \left(Cosh[h_i]\mathbb{1} + Sinh[h_i]\,L_0^{(i)}\right)\left(\mathbb{1} + a_i\,e^{-h_i}\,L_+^{(i)}\right) \end{aligned} \tag{198}$$

which follows from the identitities:

$$L_0^{(i)}\,L_0^{(i)} = \frac{1}{4}\,\mathbb{1} \tag{199}$$

$$L_+^{(i)}\,L_+^{(i)} = \mathbf{0} \tag{200}$$

$$\tag{201}$$

The explicit form of the matrices $L_a^{(i)}$ and $\mathbb{L}^{(i)}$ is given in appendix A.

We are now ready to construct the central charges and their modulus square whose minimization with respect to the fields yields the values of the fixed scalars.

Let us introduce the charge vector:

$$\vec{Q} = \begin{pmatrix} g^1 \\ g^2 \\ g^3 \\ g^4 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \tag{202}$$

Following our general formulae we can write the central charge vector as follows:

$$\vec{Z} \,=\, \mathcal{S}\,\mathbb{C}\,\prod_{i=1}^{3}\,\mathbb{L}^{(i)}(-h_1,-a_i)\,\vec{Q} \tag{203}$$

where $\mathcal{S}$ is some unitary matrix and $\mathbb{C}$ is the symplectic metric.

At this point it is immediate to write down the potential, whose minimization with respect to the scalar fields yields the values of the fixed scalars at the horizon.

We have:

$$V(\vec{Q},h,a) \equiv \vec{Z}^{\dagger}\,\vec{Z} \,=\, \vec{Q}\,\prod_{i=1}^{3}\,M^{(i)}\,(h_i,a_i)\,\vec{Q} \tag{204}$$

where:

$$M^{(i)}\,(h_i,a_i)\,\equiv\,\left[\mathbb{L}^{(i)}\,(-h_i,-a_i)\right]^{T}\,\mathbb{L}^{(i)}\,(-h_i,-a_i) \tag{205}$$

Rather then working out the derivatives of this potential and equating them to zero, we can just use the results of paper [20]. It suffices to write the correspondence between our solvable Lie algebra fields and the 3 complex scalar fields S,T,U used in the $N=2$ standard parametrization of the theory. This correspondence is:

$$\begin{aligned} T &= a_1 \,+\mathrm{i}\,\exp[2h_1] \\ U &= a_2 \,+\mathrm{i}\,\exp[2h_2] \\ S &= a_3 \,+\mathrm{i}\,\exp[2h_3] \end{aligned} \tag{206}$$

and it is established with the following argument. The symplectic section of special geometry X^{Λ} is defined, in terms of the $SO(2,2)$ coset representative $L^{\Lambda}{}_{\Sigma}(\phi)$, by the formula (see eq.(C.1) of [22]):

$$\frac{1}{\sqrt{X^{\Lambda}\,X^{\Sigma}}}\,X^{\Lambda} = \frac{1}{\sqrt{2}}\,\left(L^{\Lambda}{}_{1} + \mathrm{i}\,L^{\Lambda}{}_{2}\right) \tag{207}$$

Using for $L^{\Lambda}{}_{\Sigma}(\phi)$ the upper 4×4 block of the product $\mathbb{L}^{(1)}\,(h_1,a_1)\,\mathbb{L}^{(2)}\,(h_2,a_2)$ and using for the symplectic section X^{Λ} that given in eq.(58) of [20] we obtain the first two lines of eq.(206). The last line of the same equation is obtained by identifying the $SU(1,1)$ matrix:

$$\mathcal{C}\,\begin{pmatrix} e^{h_3} & a_3 \\ 0 & e^{-h3} \end{pmatrix}\,\mathcal{C}^{-1} \tag{208}$$

where $\mathcal{C}$ is the 2–dimensional Cayley matrix with the matrix $M(S)$ defined in eq.(3.30) of [22].

Given this identification of the fields, the fixed values at the horizon are given by eq.(37) of [20].

We can therefore conclude that we have determined the fixed values of the scalar fields at the horizon in a general $N=8$ BPS saturated black–hole.

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Appendix A

The explicit expression for the generators of $SL(2,\mathbb{R})^3$ of section 7.1 is:

$$
\begin{aligned}
L_0^{(1)} &= \begin{pmatrix}
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0
\end{pmatrix} \\
L_+^{(1)} &= \begin{pmatrix}
0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0
\end{pmatrix} \\
L_-^{(1)} &= \begin{pmatrix}
0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0
\end{pmatrix} \\
L_0^{(2)} &= \begin{pmatrix}
0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0
\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
L_+^{(2)} &= \begin{pmatrix}
0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0
\end{pmatrix} \\
L_-^{(2)} &= \begin{pmatrix}
0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0
\end{pmatrix} \\
L_0^{(3)} &= \begin{pmatrix}
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}
\end{pmatrix} \\
L_+^{(3)} &= \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \\
L_-^{(3)} &= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{aligned}
\tag{209}
$$

Furthermore, the explicit expression for the coset representatives of $\frac{SL(2,\mathbb{R})^3}{U(1)^3}$ in the same section is:

$$\mathbb{L}^{(1)}\left(h_1, a_1\right) =$$

$$\begin{pmatrix} \cosh h_1 & -a_1 & -a_1 & \sinh h_1 & 0 & 0 & 0 & 0 \\ a_1 & \cosh h_1 & -\sinh h_1 & -a_1 & 0 & 0 & 0 & 0 \\ -a_1 & -\sinh h_1 & \cosh h_1 & a_1 & 0 & 0 & 0 & 0 \\ \sinh h_1 & -a_1 & -a_1 & \cosh h_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cosh h_1 & -a_1 & a_1 & -\sinh h_1 \\ 0 & 0 & 0 & 0 & a_1 & \cosh h_1 & \sinh h_1 & a_1 \\ 0 & 0 & 0 & 0 & a_1 & \sinh h_1 & \cosh h_1 & a_1 \\ 0 & 0 & 0 & 0 & -\sinh h_1 & a_1 & -a_1 & \cosh h_1 \end{pmatrix}$$

$$\mathbb{L}^{(2)}\left(h_2, a_2\right) =$$

$$\begin{pmatrix} \cosh h_2 & -a_2 & -a_2 & -\sinh h_2 & 0 & 0 & 0 & 0 \\ a_2 & \cosh h_2 & -\sinh h_2 & a_2 & 0 & 0 & 0 & 0 \\ -a_2 & -\sinh h_2 & \cosh h_2 & -a_2 & 0 & 0 & 0 & 0 \\ -\sinh h_2 & a_2 & a_2 & \cosh h_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cosh h_2 & -a_2 & a_2 & \sinh h_2 \\ 0 & 0 & 0 & 0 & a_2 & \cosh h_2 & \sinh h_2 & -a_2 \\ 0 & 0 & 0 & 0 & a_2 & \sinh h_2 & \cosh h_2 & -a_2 \\ 0 & 0 & 0 & 0 & \sinh h_2 & -a_2 & a_2 & \cosh h_2 \end{pmatrix}$$

$$\mathbb{L}^{(3)}\left(h_3, a_3\right) =$$

$$\begin{pmatrix} 1\,e^{h_3} & 0 & 0 & 0 & 2\,a_3 & 0 & 0 & 0 \\ 0 & 1\,e^{h_3} & 0 & 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 1\,e^{h_3} & 0 & 0 & 0 & -\,a_3 & 0 \\ 0 & 0 & 0 & 1\,e^{h_3} & 0 & 0 & 0 & -\,a_3 \\ 0 & 0 & 0 & 0 & \frac{1}{e^{h_3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{e^{h_3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{e^{h_3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{e^{h_3}} \end{pmatrix} \tag{210}$$

Appendix B

In this appendix we give several tables concerning various results obtained by computer-aided computations about roots and weights and their relations to the physical fields of the solvable Lie algebra of $E_{7(7)}/SU(8)$.

Table 1: **The abelian ideals $\mathbb{D}_r^+$ and the roots of $E_{7(7)}$:**

Type IIA field	Root name	Dynkin labels		Type IIA field	Root name	Dynkin labels
			$\mathbb{D}_1^+$			
A_{10}	$\vec{\alpha}_{1,1}$	$\{0,0,0,0,0,0,1\}$				
			$\mathbb{D}_2^+$			
$B_{9,10}$	$\vec{\alpha}_{2,1}$	$\{0,0,0,0,0,1,0\}$		$g_{9,10}$	$\vec{\alpha}_{2,2}$	$\{0,0,0,0,1,0,0\}$
A_9	$\vec{\alpha}_{2,3}$	$\{0,0,0,0,0,1,1\}$				
			$\mathbb{D}_3^+$			
$B_{8,9}$	$\vec{\alpha}_{3,1}$	$\{0,0,0,1,1,1,0\}$		$g_{8,9}$	$\vec{\alpha}_{3,2}$	$\{0,0,0,1,0,0,0\}$
$B_{8,10}$	$\vec{\alpha}_{3,3}$	$\{0,0,0,1,0,1,0\}$		$g_{8,10}$	$\vec{\alpha}_{3,4}$	$\{0,0,0,1,1,0,0\}$
A_8	$\vec{\alpha}_{3,5}$	$\{0,0,0,1,1,1,1\}$		$A_{8,9,10}$	$\vec{\alpha}_{3,6}$	$\{0,0,0,1,0,1,1\}$
			$\mathbb{D}_4^+$			
$B_{7,8}$	$\vec{\alpha}_{4,1}$	$\{0,0,1,2,1,1,0\}$		$g_{7,8}$	$\vec{\alpha}_{4,2}$	$\{0,0,1,0,0,0,0\}$
$B_{7,9}$	$\vec{\alpha}_{4,3}$	$\{0,0,1,1,1,1,0\}$		$g_{7,9}$	$\vec{\alpha}_{4,4}$	$\{0,0,1,1,0,0,0\}$
$B_{7,10}$	$\vec{\alpha}_{4,5}$	$\{0,0,1,1,0,1,0\}$		$g_{7,10}$	$\vec{\alpha}_{4,6}$	$\{0,0,1,1,1,0,0\}$
$A_{7,9,10}$	$\vec{\alpha}_{4,7}$	$\{0,0,1,2,1,1,1\}$		$A_{7,8,10}$	$\vec{\alpha}_{4,8}$	$\{0,0,1,1,1,1,1\}$
$A_{7,8,9}$	$\vec{\alpha}_{4,9}$	$\{0,0,1,1,0,1,1\}$		A_7	$\vec{\alpha}_{4,10}$	$\{0,0,1,2,1,2,1\}$
			$\mathbb{D}_5^+$			
$B_{6,7}$	$\vec{\alpha}_{5,1}$	$\{0,1,2,2,1,1,0\}$		$g_{6,7}$	$\vec{\alpha}_{5,2}$	$\{0,1,0,0,0,0,0\}$
$B_{6,8}$	$\vec{\alpha}_{5,3}$	$\{0,1,1,2,1,1,0\}$		$g_{6,8}$	$\vec{\alpha}_{5,4}$	$\{0,1,1,0,0,0,0\}$
$B_{6,9}$	$\vec{\alpha}_{5,5}$	$\{0,1,1,1,1,1,0\}$		$g_{6,9}$	$\vec{\alpha}_{5,6}$	$\{0,1,1,1,0,0,0\}$
$B_{6,10}$	$\vec{\alpha}_{5,7}$	$\{0,1,1,1,0,1,0\}$		$g_{6,10}$	$\vec{\alpha}_{5,8}$	$\{0,1,1,1,1,0,0\}$
$A_{6,8,9}$	$\vec{\alpha}_{5,9}$	$\{0,1,2,2,1,1,1\}$		$A_{6,7,9}$	$\vec{\alpha}_{5,10}$	$\{0,1,1,2,1,1,1\}$
$A_{6,7,8}$	$\vec{\alpha}_{5,11}$	$\{0,1,1,1,1,1,1\}$		$A_{\mu\nu\rho}$	$\vec{\alpha}_{5,12}$	$\{0,1,1,1,0,1,1\}$
$A_{6,7,10}$	$\vec{\alpha}_{5,13}$	$\{0,1,1,2,1,2,1\}$		$A_{6,8,10}$	$\vec{\alpha}_{5,14}$	$\{0,1,2,2,1,2,1\}$
$A_{6,9,10}$	$\vec{\alpha}_{5,15}$	$\{0,1,2,3,1,2,1\}$		A_6	$\vec{\alpha}_{5,16}$	$\{0,1,2,3,2,2,1\}$
			$\mathbb{D}_6^+$			
$B_{5,6}$	$\vec{\alpha}_{6,1}$	$\{1,2,2,2,1,1,0\}$		$g_{5,6}$	$\vec{\alpha}_{6,2}$	$\{1,0,0,0,0,0,0\}$
$B_{5,7}$	$\vec{\alpha}_{6,3}$	$\{1,1,2,2,1,1,0\}$		$g_{5,7}$	$\vec{\alpha}_{6,4}$	$\{1,1,0,0,0,0,0\}$
$B_{5,8}$	$\vec{\alpha}_{6,5}$	$\{1,1,1,2,1,1,0\}$		$g_{5,8}$	$\vec{\alpha}_{6,6}$	$\{1,1,1,0,0,0,0\}$
$B_{5,9}$	$\vec{\alpha}_{6,7}$	$\{1,1,1,1,1,1,0\}$		$g_{5,9}$	$\vec{\alpha}_{6,8}$	$\{1,1,1,1,0,0,0\}$
$B_{5,10}$	$\vec{\alpha}_{6,9}$	$\{1,1,1,1,0,1,0\}$		$g_{5,10}$	$\vec{\alpha}_{6,10}$	$\{1,1,1,1,1,0,0\}$
$B_{\mu\nu}$	$\vec{\alpha}_{6,11}$	$\{1,2,3,4,2,3,2\}$		A_5	$\vec{\alpha}_{6,12}$	$\{1,2,3,4,2,3,1\}$
$A_{\mu\nu 6}$	$\vec{\alpha}_{6,13}$	$\{1,2,2,2,1,1,1\}$		$A_{\mu\nu 7}$	$\vec{\alpha}_{6,14}$	$\{1,1,2,2,1,1,1\}$
$A_{\mu\nu 8}$	$\vec{\alpha}_{6,15}$	$\{1,1,1,2,1,1,1\}$		$A_{\mu\nu 9}$	$\vec{\alpha}_{6,16}$	$\{1,1,1,1,1,1,1\}$
$A_{\mu\nu 10}$	$\vec{\alpha}_{6,17}$	$\{1,1,1,1,0,1,1\}$		$A_{5,6,7}$	$\vec{\alpha}_{6,18}$	$\{1,1,1,2,1,2,1\}$
$A_{5,6,8}$	$\vec{\alpha}_{6,19}$	$\{1,1,2,2,1,2,1\}$		$A_{5,6,9}$	$\vec{\alpha}_{6,20}$	$\{1,1,2,3,1,2,1\}$
$A_{5,6,10}$	$\vec{\alpha}_{6,21}$	$\{1,1,2,3,2,2,1\}$		$A_{5,7,8}$	$\vec{\alpha}_{6,22}$	$\{1,2,2,2,1,2,1\}$
$A_{5,7,9}$	$\vec{\alpha}_{6,23}$	$\{1,2,2,3,1,2,1\}$		$A_{5,7,10}$	$\vec{\alpha}_{6,24}$	$\{1,2,2,3,2,2,1\}$
$A_{5,8,9}$	$\vec{\alpha}_{6,25}$	$\{1,2,3,3,1,2,1\}$		$A_{5,8,10}$	$\vec{\alpha}_{6,26}$	$\{1,2,3,3,2,2,1\}$
$A_{5,9,10}$	$\vec{\alpha}_{6,27}$	$\{1,2,3,4,2,2,1\}$				

Table 2: **Weights of the 56 representation of $E_{7(7)}$:**

Weight name	q^ℓ vector		Weight name	q^ℓ vector
$\vec{W}^{(1)}$ =	$\{2,3,4,5,3,3,1\}$		$\vec{W}^{(2)}$ =	$\{2,2,2,2,1,1,1\}$
$\vec{W}^{(3)}$ =	$\{1,2,2,2,1,1,1\}$		$\vec{W}^{(4)}$ =	$\{1,1,2,2,1,1,1\}$
$\vec{W}^{(5)}$ =	$\{1,1,1,2,1,1,1\}$		$\vec{W}^{(6)}$ =	$\{1,1,1,1,1,1,1\}$
$\vec{W}^{(7)}$ =	$\{2,3,3,3,1,2,1\}$		$\vec{W}^{(8)}$ =	$\{2,2,3,3,1,2,1\}$
$\vec{W}^{(9)}$ =	$\{2,2,2,3,1,2,1\}$		$\vec{W}^{(10)}$ =	$\{2,2,2,2,1,2,1\}$
$\vec{W}^{(11)}$ =	$\{1,2,2,2,1,2,1\}$		$\vec{W}^{(12)}$ =	$\{1,1,2,2,1,2,1\}$
$\vec{W}^{(13)}$ =	$\{1,1,1,2,1,2,1\}$		$\vec{W}^{(14)}$ =	$\{1,2,2,3,1,2,1\}$
$\vec{W}^{(15)}$ =	$\{1,2,3,3,1,2,1\}$		$\vec{W}^{(16)}$ =	$\{1,1,2,3,1,2,1\}$
$\vec{W}^{(17)}$ =	$\{2,2,2,2,1,1,0\}$		$\vec{W}^{(18)}$ =	$\{1,2,2,2,1,1,0\}$
$\vec{W}^{(19)}$ =	$\{1,1,2,2,1,1,0\}$		$\vec{W}^{(20)}$ =	$\{1,1,1,2,1,1,0\}$
$\vec{W}^{(21)}$ =	$\{1,1,1,1,1,1,0\}$		$\vec{W}^{(22)}$ =	$\{1,1,1,1,1,0,0\}$
$\vec{W}^{(23)}$ =	$\{3,4,5,6,3,4,2\}$		$\vec{W}^{(24)}$ =	$\{2,4,5,6,3,4,2\}$
$\vec{W}^{(25)}$ =	$\{2,3,5,6,3,4,2\}$		$\vec{W}^{(26)}$ =	$\{2,3,4,6,3,4,2\}$
$\vec{W}^{(27)}$ =	$\{2,3,4,5,3,4,2\}$		$\vec{W}^{(28)}$ =	$\{2,3,4,5,3,3,2\}$
$\vec{W}^{(29)}$ =	$\{1,1,1,1,0,1,1\}$		$\vec{W}^{(30)}$ =	$\{1,2,3,4,2,3,1\}$
$\vec{W}^{(31)}$ =	$\{2,2,3,4,2,3,1\}$		$\vec{W}^{(32)}$ =	$\{2,3,3,4,2,3,1\}$
$\vec{W}^{(33)}$ =	$\{2,3,4,4,2,3,1\}$		$\vec{W}^{(34)}$ =	$\{2,3,4,5,2,3,1\}$
$\vec{W}^{(35)}$ =	$\{1,1,2,3,2,2,1\}$		$\vec{W}^{(36)}$ =	$\{1,2,2,3,2,2,1\}$
$\vec{W}^{(37)}$ =	$\{1,2,3,3,2,2,1\}$		$\vec{W}^{(38)}$ =	$\{1,2,3,4,2,2,1\}$
$\vec{W}^{(39)}$ =	$\{2,2,3,4,2,2,1\}$		$\vec{W}^{(40)}$ =	$\{2,3,3,4,2,2,1\}$
$\vec{W}^{(41)}$ =	$\{2,3,4,4,2,2,1\}$		$\vec{W}^{(42)}$ =	$\{2,2,3,3,2,2,1\}$
$\vec{W}^{(43)}$ =	$\{2,2,2,3,2,2,1\}$		$\vec{W}^{(44)}$ =	$\{2,3,3,3,2,2,1\}$
$\vec{W}^{(45)}$ =	$\{1,2,3,4,2,3,2\}$		$\vec{W}^{(46)}$ =	$\{2,2,3,4,2,3,2\}$
$\vec{W}^{(47)}$ =	$\{2,3,3,4,2,3,2\}$		$\vec{W}^{(48)}$ =	$\{2,3,4,4,2,3,2\}$
$\vec{W}^{(49)}$ =	$\{2,3,4,5,2,3,2\}$		$\vec{W}^{(50)}$ =	$\{2,3,4,5,2,4,2\}$
$\vec{W}^{(51)}$ =	$\{0,0,0,0,0,0,0\}$		$\vec{W}^{(52)}$ =	$\{1,0,0,0,0,0,0\}$
$\vec{W}^{(53)}$ =	$\{1,1,0,0,0,0,0\}$		$\vec{W}^{(54)}$ =	$\{1,1,1,0,0,0,0\}$
$\vec{W}^{(55)}$ =	$\{1,1,1,1,0,0,0\}$		$\vec{W}^{(56)}$ =	$\{1,1,1,1,0,1,0\}$

Table 3: **Scalar products of weights and Cartan dilatons**:

$\vec{\Lambda}^{(1)} \cdot \vec{h} = \frac{-h_1-h_2-h_3-h_4-h_5+h_6}{\sqrt{6}}$	$\vec{\Lambda}^{(2)} \cdot \vec{h} = \frac{-h_1+h_2+h_3+h_4+h_5+h_6}{\sqrt{6}}$
$\vec{\Lambda}^{(3)} \cdot \vec{h} = \frac{h_1-h_2+h_3+h_4+h_5+h_6}{\sqrt{6}}$	$\vec{\Lambda}^{(4)} \cdot \vec{h} = \frac{h_1+h_2-h_3+h_4+h_5+h_6}{\sqrt{6}}$
$\vec{\Lambda}^{(5)} \cdot \vec{h} = \frac{h_1+h_2+h_3-h_4+h_5+h_6}{\sqrt{6}}$	$\vec{\Lambda}^{(6)} \cdot \vec{h} = \frac{h_1+h_2+h_3+h_4-h_5+h_6}{\sqrt{6}}$
$\vec{\Lambda}^{(7)} \cdot \vec{h} = \frac{-h_1-h_2+h_3+h_4+h_5-h_6}{\sqrt{6}}$	$\vec{\Lambda}^{(8)} \cdot \vec{h} = \frac{-h_1+h_2-h_3+h_4+h_5-h_6}{\sqrt{6}}$
$\vec{\Lambda}^{(9)} \cdot \vec{h} = \frac{-h_1+h_2+h_3-h_4+h_5-h_6}{\sqrt{6}}$	$\vec{\Lambda}^{(10)} \cdot \vec{h} = \frac{-h_1+h_2+h_3+h_4-h_5-h_6}{\sqrt{6}}$
$\vec{\Lambda}^{(11)} \cdot \vec{h} = \frac{h_1-h_2+h_3+h_4-h_5-h_6}{\sqrt{6}}$	$\vec{\Lambda}^{(12)} \cdot \vec{h} = \frac{h_1+h_2-h_3+h_4-h_5-h_6}{\sqrt{6}}$
$\vec{\Lambda}^{(13)} \cdot \vec{h} = \frac{h_1+h_2+h_3-h_4-h_5-h_6}{\sqrt{6}}$	$\vec{\Lambda}^{(14)} \cdot \vec{h} = \frac{h_1-h_2+h_3-h_4+h_5-h_6}{\sqrt{6}}$
$\vec{\Lambda}^{(15)} \cdot \vec{h} = \frac{h_1-h_2-h_3+h_4+h_5-h_6}{\sqrt{6}}$	$\vec{\Lambda}^{(16)} \cdot \vec{h} = \frac{h_1+h_2-h_3-h_4+h_5-h_6}{\sqrt{6}}$
$\vec{\Lambda}^{(17)} \cdot \vec{h} = \frac{-\left(\sqrt{2}\,h_1\right)+h_7}{\sqrt{3}}$	$\vec{\Lambda}^{(18)} \cdot \vec{h} = \frac{-\left(\sqrt{2}\,h_2\right)+h_7}{\sqrt{3}}$
$\vec{\Lambda}^{(19)} \cdot \vec{h} = \frac{-\left(\sqrt{2}\,h_3\right)+h_7}{\sqrt{3}}$	$\vec{\Lambda}^{(20)} \cdot \vec{h} = \frac{-\left(\sqrt{2}\,h_4\right)+h_7}{\sqrt{3}}$
$\vec{\Lambda}^{(21)} \cdot \vec{h} = \frac{-\left(\sqrt{2}\,h_5\right)+h_7}{\sqrt{3}}$	$\vec{\Lambda}^{(22)} \cdot \vec{h} = \frac{\sqrt{2}\,h_6+h_7}{\sqrt{3}}$
$\vec{\Lambda}^{(23)} \cdot \vec{h} = -\frac{\sqrt{2}\,h_1+h_7}{\sqrt{3}}$	$\vec{\Lambda}^{(24)} \cdot \vec{h} = -\frac{\sqrt{2}\,h_2+h_7}{\sqrt{3}}$
$\vec{\Lambda}^{(25)} \cdot \vec{h} = -\frac{\sqrt{2}\,h_3+h_7}{\sqrt{3}}$	$\vec{\Lambda}^{(26)} \cdot \vec{h} = -\frac{\sqrt{2}\,h_4+h_7}{\sqrt{3}}$
$\vec{\Lambda}^{(27)} \cdot \vec{h} = -\frac{\sqrt{2}\,h_5+h_7}{\sqrt{3}}$	$\vec{\Lambda}^{(28)} \cdot \vec{h} = \frac{\sqrt{2}\,h_6-h_7}{\sqrt{3}}$

Table 4: **The step operators of the $SU(8)$ subalgebra of $E_{7(7)}$:**

#	Root name	Root vector	SU(8) step operator in terms of $E_{7(7)}$ step oper.
1	$\vec{a}_1$	$\{-1,-1,1,1,0,0,0\}$	$\begin{cases} X^{a_1} = 2\left(B^{\alpha_{3,3}} + B^{\alpha_{3,4}} - B^{\alpha_{4,3}} + B^{\alpha_{4,4}}\right) \\ Y^{a_1} = 2\left(B^{\alpha_{3,1}} - B^{\alpha_{3,2}} + B^{\alpha_{4,5}} + B^{\alpha_{4,6}}\right) \end{cases}$
2	$\vec{a}_2$	$\{0,0,-1,-1,1,1,0\}$	$\begin{cases} X^{a_2} = 2\left(B^{\alpha_{5,3}} + B^{\alpha_{5,4}} - B^{\alpha_{6,3}} + B^{\alpha_{6,4}}\right) \\ Y^{a_2} = 2\left(B^{\alpha_{5,1}} - B^{\alpha_{5,2}} + B^{\alpha_{6,5}} + B^{\alpha_{6,6}}\right) \end{cases}$
3	$\vec{a}_3$	$\{1,1,1,1,0,0,0\}$	$\begin{cases} X^{a_3} = 2\left(B^{\alpha_{3,3}} + B^{\alpha_{3,4}} + B^{\alpha_{4,3}} - B^{\alpha_{4,4}}\right) \\ Y^{a_3} = 2\left(-B^{\alpha_{3,1}} + B^{\alpha_{3,2}} + B^{\alpha_{4,5}} + B^{\alpha_{4,6}}\right) \end{cases}$
4	$\vec{a}_4$	$\{-1,0,-1,0,-1,0,-1\}$	$\begin{cases} X^{a_4} = 2\left(B^{\alpha_{2,3}} + B^{\alpha_{4,7}} - B^{\alpha_{6,12}} + B^{\alpha_{6,13}}\right) \\ Y^{a_4} = 2\left(B^{\alpha_{1,1}} + B^{\alpha_{4,10}} + B^{\alpha_{6,22}} + B^{\alpha_{6,27}}\right) \end{cases}$
5	$\vec{a}_5$	$\{1,-1,0,0,1,-1,0\}$	$\begin{cases} X^{a_5} = 2\left(B^{\alpha_{5,5}} + B^{\alpha_{5,6}} - B^{\alpha_{6,9}} + B^{\alpha_{6,10}}\right) \\ Y^{a_5} = -2\left(B^{\alpha_{5,7}} - B^{\alpha_{5,8}} + B^{\alpha_{6,7}} + B^{\alpha_{6,8}}\right) \end{cases}$
6	$\vec{a}_6$	$\{0,0,1,-1,-1,1,0\}$	$\begin{cases} X^{a_6} = 2\left(-B^{\alpha_{5,1}} - B^{\alpha_{5,2}} - B^{\alpha_{6,5}} + B^{\alpha_{6,6}}\right) \\ Y^{a_6} = -2\left(-B^{\alpha_{5,3}} + B^{\alpha_{5,4}} + B^{\alpha_{6,3}} + B^{\alpha_{6,4}}\right) \end{cases}$
7	$\vec{a}_7$	$\{-1,1,0,0,1,-1,0\}$	$\begin{cases} X^{a_7} = 2\left(B^{\alpha_{5,5}} + B^{\alpha_{5,6}} + B^{\alpha_{6,9}} - B^{\alpha_{6,10}}\right) \\ Y^{a_7} = -2\left(-B^{\alpha_{5,7}} + B^{\alpha_{5,8}} + B^{\alpha_{6,7}} + B^{\alpha_{6,8}}\right) \end{cases}$
8	$\vec{a}_{12}$	$\{-1,-1,0,0,1,1,0\}$	$\begin{cases} X^{a_{12}} = 2\left(B^{\alpha_{5,7}} + B^{\alpha_{5,8}} - B^{\alpha_{6,7}} + B^{\alpha_{6,8}}\right) \\ Y^{a_{12}} = 2\left(B^{\alpha_{5,5}} - B^{\alpha_{5,6}} + B^{\alpha_{6,9}} + B^{\alpha_{6,10}}\right) \end{cases}$
9	$\vec{a}_{23}$	$\{1,1,0,0,1,1,0\}$	$\begin{cases} X^{a_{23}} = 2\left(B^{\alpha_{5,7}} + B^{\alpha_{5,8}} + B^{\alpha_{6,7}} - B^{\alpha_{6,8}}\right) \\ Y^{a_{23}} = 2\left(-B^{\alpha_{5,5}} + B^{\alpha_{5,6}} + B^{\alpha_{6,9}} + B^{\alpha_{6,10}}\right) \end{cases}$
10	$\vec{a}_{34}$	$\{0,1,0,1,-1,0,-1\}$	$\begin{cases} X^{a_{34}} = 2\left(B^{\alpha_{3,6}} + B^{\alpha_{4,8}} - B^{\alpha_{6,24}} - B^{\alpha_{6,25}}\right) \\ Y^{a_{34}} = 2\left(-B^{\alpha_{3,5}} + B^{\alpha_{4,9}} - B^{\alpha_{6,23}} + B^{\alpha_{6,26}}\right) \end{cases}$
11	$\vec{a}_{45}$	$\{0,-1,-1,0,0,-1,-1\}$	$\begin{cases} X^{a_{45}} = 2\left(B^{\alpha_{5,11}} + B^{\alpha_{5,15}} - B^{\alpha_{6,17}} + B^{\alpha_{6,21}}\right) \\ Y^{a_{45}} = -2\left(B^{\alpha_{5,12}} - B^{\alpha_{5,16}} + B^{\alpha_{6,16}} + B^{\alpha_{6,20}}\right) \end{cases}$

Table 5: **The step operators of the** $SU(8)$**...***continued* 2^{nd}:

#	Root name	Root vector	SU(8) step operator in terms of $E_{7(7)}$ step oper.
12	$\vec{a}_{56}$	$\{1,-1,1,-1,0,0,0\}$	$\begin{cases} X^{a_{56}} = 2\left(B^{\alpha_{3,1}} + B^{\alpha_{3,2}} - B^{\alpha_{4,5}} + B^{\alpha_{4,6}}\right) \\ Y^{a_{56}} = -2\left(B^{\alpha_{3,3}} - B^{\alpha_{3,4}} + B^{\alpha_{4,3}} + B^{\alpha_{4,4}}\right) \end{cases}$
13	$\vec{a}_{67}$	$\{-1,1,1,-1,0,0,0\}$	$\begin{cases} X^{a_{67}} = 2\left(B^{\alpha_{3,1}} + B^{\alpha_{3,2}} + B^{\alpha_{4,5}} - B^{\alpha_{4,6}}\right) \\ Y^{a_{67}} = -2\left(-B^{\alpha_{3,3}} + B^{\alpha_{3,4}} + B^{\alpha_{4,3}} + B^{\alpha_{4,4}}\right) \end{cases}$
14	$\vec{a}_{123}$	$\{0,0,1,1,1,1,0\}$	$\begin{cases} X^{a_{123}} = 2\left(B^{\alpha_{5,3}} + B^{\alpha_{5,4}} + B^{\alpha_{6,3}} - B^{\alpha_{6,4}}\right) \\ Y^{a_{123}} = 2\left(-B^{\alpha_{5,1}} + B^{\alpha_{5,2}} + B^{\alpha_{6,5}} + B^{\alpha_{6,6}}\right) \end{cases}$
15	$\vec{a}_{234}$	$\{0,1,-1,0,0,1,-1\}$	$\begin{cases} X^{a_{234}} = 2\left(B^{\alpha_{5,12}} + B^{\alpha_{5,16}} + B^{\alpha_{6,16}} - B^{\alpha_{6,20}}\right) \\ Y^{a_{234}} = 2\left(-B^{\alpha_{5,11}} + B^{\alpha_{5,15}} + B^{\alpha_{6,17}} + B^{\alpha_{6,21}}\right) \end{cases}$
16	$\vec{a}_{345}$	$\{1,0,0,1,0,-1,-1\}$	$\begin{cases} X^{a_{345}} = 2\left(B^{\alpha_{5,9}} + B^{\alpha_{5,13}} + B^{\alpha_{6,15}} - B^{\alpha_{6,19}}\right) \\ Y^{a_{345}} = -2\left(-B^{\alpha_{5,10}} + B^{\alpha_{5,14}} + B^{\alpha_{6,14}} + B^{\alpha_{6,18}}\right) \end{cases}$
17	$\vec{a}_{456}$	$\{0,-1,0,-1,-1,0,-1\}$	$\begin{cases} X^{a_{456}} = 2\left(B^{\alpha_{3,6}} + B^{\alpha_{4,8}} + B^{\alpha_{6,24}} + B^{\alpha_{6,25}}\right) \\ Y^{a_{456}} = 2\left(B^{\alpha_{3,5}} - B^{\alpha_{4,9}} - B^{\alpha_{6,23}} + B^{\alpha_{6,26}}\right) \end{cases}$
18	$\vec{a}_{567}$	$\{0,0,1,-1,1,-1,0\}$	$\begin{cases} X^{a_{567}} = 2\left(B^{\alpha_{5,1}} + B^{\alpha_{5,2}} - B^{\alpha_{6,5}} + B^{\alpha_{6,6}}\right) \\ Y^{a_{567}} = -2\left(B^{\alpha_{5,3}} - B^{\alpha_{5,4}} + B^{\alpha_{6,3}} + B^{\alpha_{6,4}}\right) \end{cases}$

Table 6: **The step operators of the** $SU(8)$**...***continued* 3^{rd}:

#	Root name	Root vector	SU(8) step operator in terms of $E_{7(7)}$ step oper.
19	$\vec{a}_{1234}$	$\{-1,0,0,1,0,1,-1\}$	$\begin{cases} X^{a_{1234}} = 2\left(B^{\alpha_{5,10}} + B^{\alpha_{5,14}} + B^{\alpha_{6,14}} - B^{\alpha_{6,18}}\right) \\ Y^{a_{1234}} = 2\left(-B^{\alpha_{5,9}} + B^{\alpha_{5,13}} + B^{\alpha_{6,15}} + B^{\alpha_{6,19}}\right) \end{cases}$
20	$\vec{a}_{2345}$	$\{1,0,-1,0,1,0,-1\}$	$\begin{cases} X^{a_{2345}} = 2\left(-B^{\alpha_{1,1}} + B^{\alpha_{4,10}} - B^{\alpha_{6,22}} + B^{\alpha_{6,27}}\right) \\ Y^{a_{2345}} = -2\left(B^{\alpha_{2,3}} - B^{\alpha_{4,7}} + B^{\alpha_{6,12}} + B^{\alpha_{6,13}}\right) \end{cases}$
21	$\vec{a}_{3456}$	$\{1,0,1,0,-1,0,-1\}$	$\begin{cases} X^{a_{3456}} = 2\left(B^{\alpha_{2,3}} + B^{\alpha_{4,7}} + B^{\alpha_{6,12}} - B^{\alpha_{6,13}}\right) \\ Y^{a_{3456}} = 2\left(-B^{\alpha_{1,1}} - B^{\alpha_{4,10}} + B^{\alpha_{6,22}} + B^{\alpha_{6,27}}\right) \end{cases}$
22	$\vec{a}_{4567}$	$\{-1,0,0,-1,0,-1,-1\}$	$\begin{cases} X^{a_{4567}} = 2\left(B^{\alpha_{5,9}} + B^{\alpha_{5,13}} - B^{\alpha_{6,15}} + B^{\alpha_{6,19}}\right) \\ Y^{a_{4567}} = -2\left(B^{\alpha_{5,10}} - B^{\alpha_{5,14}} + B^{\alpha_{6,14}} + B^{\alpha_{6,18}}\right) \end{cases}$
23	$\vec{a}_{12345}$	$\{0,-1,0,1,1,0,-1\}$	$\begin{cases} X^{a_{12345}} = 2\left(B^{\alpha_{3,5}} + B^{\alpha_{4,9}} + B^{\alpha_{6,23}} + B^{\alpha_{6,26}}\right) \\ Y^{a_{12345}} = -2\left(B^{\alpha_{3,6}} - B^{\alpha_{4,8}} - B^{\alpha_{6,24}} + B^{\alpha_{6,25}}\right) \end{cases}$
24	$\vec{a}_{23456}$	$\{1,0,0,-1,0,1,-1\}$	$\begin{cases} X^{a_{23456}} = 2\left(B^{\alpha_{5,10}} + B^{\alpha_{5,14}} - B^{\alpha_{6,14}} + B^{\alpha_{6,18}}\right) \\ Y^{a_{23456}} = 2\left(B^{\alpha_{5,9}} - B^{\alpha_{5,13}} + B^{\alpha_{6,15}} + B^{\alpha_{6,19}}\right) \end{cases}$
25	$\vec{a}_{34567}$	$\{0,1,1,0,0,-1,-1\}$	$\begin{cases} X^{a_{34567}} = 2\left(B^{\alpha_{5,11}} + B^{\alpha_{5,15}} + B^{\alpha_{6,17}} - B^{\alpha_{6,21}}\right) \\ Y^{a_{34567}} = -2\left(-B^{\alpha_{5,12}} + B^{\alpha_{5,16}} + B^{\alpha_{6,16}} + B^{\alpha_{6,20}}\right) \end{cases}$
26	$\vec{a}_{123456}$	$\{0,-1,1,0,0,1,-1\}$	$\begin{cases} X^{a_{123456}} = 2\left(B^{\alpha_{5,12}} + B^{\alpha_{5,16}} - B^{\alpha_{6,16}} + B^{\alpha_{6,20}}\right) \\ Y^{a_{123456}} = 2\left(B^{\alpha_{5,11}} - B^{\alpha_{5,15}} + B^{\alpha_{6,17}} + B^{\alpha_{6,21}}\right) \end{cases}$
27	$\vec{a}_{234567}$	$\{0,1,0,-1,1,0,-1\}$	$\begin{cases} X^{a_{234567}} = 2\left(-B^{\alpha_{3,5}} - B^{\alpha_{4,9}} + B^{\alpha_{6,23}} + B^{\alpha_{6,26}}\right) \\ Y^{a_{234567}} = 2\left(-B^{\alpha_{3,6}} + B^{\alpha_{4,8}} - B^{\alpha_{6,24}} + B^{\alpha_{6,25}}\right) \end{cases}$
28	$\vec{a}_{1234567}$	$\{-1,0,1,0,1,0,-1\}$	$\begin{cases} X^{a_{1234567}} = 2\left(-B^{\alpha_{1,1}} + B^{\alpha_{4,10}} + B^{\alpha_{6,22}} - B^{\alpha_{6,27}}\right) \\ Y^{a_{1234567}} = -2\left(-B^{\alpha_{2,3}} + B^{\alpha_{4,7}} + B^{\alpha_{6,12}} + B^{\alpha_{6,13}}\right) \end{cases}$

Table 7: **Weights of the** 28 **representation of** $SU(8)$:

Weight name	Weight vector		Weight name	Weight vector
		(1,1,1)		
$\vec{\Lambda}'^{(1)}$ =	$\{1,2,2,2,2,2,1\}$			
		(1,1,15)		
$\vec{\Lambda}'^{(2)}$ =	$\{0,0,0,1,1,2,1\}$		$\vec{\Lambda}'^{(3)}$ =	$\{0,0,0,1,1,1,1\}$
$\vec{\Lambda}'^{(4)}$ =	$\{0,0,0,1,1,1,0\}$		$\vec{\Lambda}'^{(5)}$ =	$\{0,0,1,2,2,2,1\}$
$\vec{\Lambda}'^{(6)}$ =	$\{0,0,0,1,2,2,1\}$		$\vec{\Lambda}'^{(7)}$ =	$\{0,0,0,0,0,1,0\}$
$\vec{\Lambda}'^{(8)}$ =	$\{0,0,0,0,0,0,0\}$		$\vec{\Lambda}'^{(9)}$ =	$\{0,0,0,0,0,1,1\}$
$\vec{\Lambda}'^{(10)}$ =	$\{0,0,1,1,1,2,1\}$		$\vec{\Lambda}'^{(11)}$ =	$\{0,0,1,1,1,1,1\}$
$\vec{\Lambda}'^{(12)}$ =	$\{0,0,1,1,1,1,0\}$		$\vec{\Lambda}'^{(13)}$ =	$\{0,0,0,0,1,1,1\}$
$\vec{\Lambda}'^{(14)}$ =	$\{0,0,0,0,1,2,1\}$		$\vec{\Lambda}'^{(15)}$ =	$\{0,0,0,0,1,1,0\}$
$\vec{\Lambda}'^{(16)}$ =	$\{0,0,1,1,2,2,1\}$			
		(1,2,6)		
$\vec{\Lambda}'^{(17)}$ =	$\{1,1,1,2,2,2,1\}$		$\vec{\Lambda}'^{(18)}$ =	$\{0,1,1,2,2,2,1\}$
$\vec{\Lambda}'^{(19)}$ =	$\{1,1,1,1,1,2,1\}$		$\vec{\Lambda}'^{(20)}$ =	$\{1,1,1,1,1,1,1\}$
$\vec{\Lambda}'^{(21)}$ =	$\{1,1,1,1,1,1,0\}$		$\vec{\Lambda}'^{(22)}$ =	$\{0,1,1,1,1,2,1\}$
$\vec{\Lambda}'^{(23)}$ =	$\{0,1,1,1,1,1,0\}$		$\vec{\Lambda}'^{(24)}$ =	$\{0,1,1,1,1,1,1\}$
$\vec{\Lambda}'^{(25)}$ =	$\{1,1,2,2,2,2,1\}$		$\vec{\Lambda}'^{(26)}$ =	$\{0,1,2,2,2,2,1\}$
$\vec{\Lambda}'^{(27)}$ =	$\{1,1,1,1,2,2,1\}$		$\vec{\Lambda}'^{(28)}$ =	$\{0,1,1,1,2,2,1\}$

Table 8: **Weights of the $\overline{2}8$ representation of $SU(8)$:**

Weight name	Weight vector		Weight name	Weight vector
		$\overline{(\mathbf{1},\mathbf{1},\mathbf{1})}$		
$-\vec{\Lambda}'^{(1)}$ =	$\{0,0,0,0,0,0,0\}$			
		$\overline{(\mathbf{1},\mathbf{1},\mathbf{15})}$		
$-\vec{\Lambda}'^{(2)}$ =	$\{1,2,2,1,1,0,0\}$		$-\vec{\Lambda}'^{(3)}$ =	$\{1,2,2,1,1,1,0\}$
$-\vec{\Lambda}'^{(4)}$ =	$\{1,2,2,1,1,1,1\}$		$-\vec{\Lambda}'^{(5)}$ =	$\{1,2,1,0,0,0,0\}$
$-\vec{\Lambda}'^{(6)}$ =	$\{1,2,2,1,0,0,0\}$		$-\vec{\Lambda}'^{(7)}$ =	$\{1,2,2,2,2,1,1\}$
$-\vec{\Lambda}'^{(8)}$ =	$\{1,2,2,2,2,2,1\}$		$-\vec{\Lambda}'^{(9)}$ =	$\{1,2,2,2,2,1,0\}$
$-\vec{\Lambda}'^{(10)}$ =	$\{1,2,1,1,1,0,0\}$		$-\vec{\Lambda}'^{(11)}$ =	$\{1,2,1,1,1,1,0\}$
$-\vec{\Lambda}'^{(12)}$ =	$\{1,2,1,1,1,1,1\}$		$-\vec{\Lambda}'^{(13)}$ =	$\{1,2,2,2,1,1,0\}$
$-\vec{\Lambda}'^{(14)}$ =	$\{1,2,2,2,1,0,0\}$		$-\vec{\Lambda}'^{(15)}$ =	$\{1,2,2,2,1,1,1\}$
$-\vec{\Lambda}'^{(16)}$ =	$\{1,2,1,1,0,0,0\}$			
		$\overline{(\mathbf{1},\mathbf{2},\mathbf{6})}$		
$-\vec{\Lambda}'^{(17)}$ =	$\{0,1,1,0,0,0,0\}$		$-\vec{\Lambda}'^{(18)}$ =	$\{1,1,1,0,0,0,0\}$
$-\vec{\Lambda}'^{(19)}$ =	$\{0,1,1,1,1,0,0\}$		$-\vec{\Lambda}'^{(20)}$ =	$\{0,1,1,1,1,1,0\}$
$-\vec{\Lambda}'^{(21)}$ =	$\{0,1,1,1,1,1,1\}$		$-\vec{\Lambda}'^{(22)}$ =	$\{1,1,1,1,1,0,0\}$
$-\vec{\Lambda}'^{(23)}$ =	$\{1,1,1,1,1,1,1\}$		$-\vec{\Lambda}'^{(24)}$ =	$\{1,1,1,1,1,1,0\}$
$-\vec{\Lambda}'^{(25)}$ =	$\{0,1,0,0,0,0,0\}$		$-\vec{\Lambda}'^{(26)}$ =	$\{1,1,0,0,0,0,0\}$
$-\vec{\Lambda}'^{(27)}$ =	$\{0,1,1,1,0,0,0\}$		$-\vec{\Lambda}'^{(28)}$ =	$\{1,1,1,1,0,0,0\}$